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New Tan hyperbolic function Solutions for the conformable time-fractional on Hirota-Satsuma equations

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***Abstract*— In this work, we present new exact solutions of the time fractional Hirota-Satsuma equations. The modified $\tanh$ expansion method is used to solve the governing equations. This method allows us to construct new families of solutions including travelling wave solutions, solitary wave solutions and periodic solutions. The analysis shows that when the value of memory index (time-fractional order) of Hirota-Satsuma equation is almost zero, the results distinct and create a wave-like configuration.**

Keywords—Homogeneous balance; Hyperbolic function solution; Travelling wave solutions; Conformablefractional derivatives; Hirota-sutuma equations.

Mathematics Subject Classifications Primary: 35C07, 49K15, 35R11, 35K99.

I. Introduction (*Heading 1*)

Fractional order differential equations describe best physical conditions and also it deals a great importance in many sciences and engineering fields. In applied physics and mathematics, nonlinear partial differential equations (NPDEs) have a vital place. These equations are mathematical models of physical occurrence that stand up in engineering, chemistry, biology, fluid dynamics, aerodynamics and physics. It is important to have information about the solutions of mathematical models. Thus, it has a vital place to obtain the analytic results of nonlinear partial differential equations in practical sciences. In the recent years, it has become gorgeous solving these equations. Therefore, some methods have been established by sciences. Some of them are: Hirota method [1], Homogeneous balance method [14], Backlund transformation [2], Burger's KdV method [3, 8], Enhanced $\left(\frac{G'}{G}\right)$-expansion method [4, 5], $\left(\frac{G'}{G}\right)$-expansion method [8, 10, 11], $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method [9], the exp(-Φ(ξ))-expansion method [6], modified generalized Riccati equation method [7], the improved F-expansion method [13, 20], the $\tan(F(\xi)/2)$ -expansion method [18], for the traveling wave solutions of the variant Bussinesq equations [12, 21, 22, 23]. Recently, [19] observed that the solutions of the modified Hirota-Satsuma system bifurcate and produce a wave pattern full memory and the pattern vanishes.

Whereas, many researchers utilized the fractional derivative [15, 16, 17] and other well-known fractional derivatives alongside some analytical methods to solve many solitary wave problems such as the Hirota-Satsuma equations [11].

$$u_t + u_{xxx} + 6uu_x - 6vv_x = 0$$
$$v_t - 2v_{xxx} - 6uv_x = 0 \tag{1}$$

However, in this article, the time fractional Hirota-Satsuma equations would be considered in the sense of the newly devised conformable fractional derivative definition. The considered problem would then be solved by employing the modified extended *tanh* expansion method after recasting the problem to an ordinary differentiation equation via wave transformation. The new analytical solutions achieved in this article are considered by hyperbolic function, trigonometric functions and rational function. The Maple software would be used in the solution of the system of algebraic equations obtained and also in the graphical illustrations of the solution.

II. Description of Conformable Derivative

A. *Definition (i): Conformable derivative*

Various exact hyperbolic function solutions for the time fractional variant bussinesq equations are constructed using the modified extended $\tanh$ expansion method in the sense of the newly devised fractional derivative called the conformable fractional derivative by [12].

Let $u:[0,\infty)\to\Re$ be a function. The α's order conformable derivative of u is defined by

$$D_t^{\alpha}u(t)=\lim_{\varepsilon\to 0}\frac{u(t+\varepsilon t^{1-\alpha})-u(t)}{\varepsilon} \quad (2)$$

for all $t>0$ and $\alpha\in(0,1)$. See also [15].

B. Definition (ii): Conformable derivative

Let $\alpha\in[0,1]$. A differential operator D^{α} is conformable if and only if D_t^0 is the identity operator and D_t^1 is the classical differential operator. Specifically, D_t^{α} is conformable if and only if for a differentiable function $u=u(t)$, $D_t^0u(t)=u(t)$ and $D_t^1u(t)=\frac{d}{dt}u(t)=u'(t)$.

Note that under this definition the operator given via (ii) is not conformable.

Further, the following theorem gives some properties of conformable derivative:

Theorem (i): Let $\alpha\in(0,1)$ and suppose $u(t)$ and $v(t)$ are $\alpha-$differentiable at $t>0$.

Then

a) $D_t^{\alpha}(t^p)=pt^{p-\alpha}$, for all $p\in\Re$.

b) $D_t^{\alpha}(q)=0$, for all constant function $u(t)=q$.

c) $D_t^{\alpha}(qu(t))=qD_t^{\alpha}(u(t))$, for all q constant. i)

d) $D_t^{\alpha}(qu(t)+rv(t))=qD_t^{\alpha}(u(t)+rD_t^{\alpha}(v(t))$, for all $q,r\in\Re$.

e) $D_t^{\alpha}(u(t)v(t))=v(t)D_t^{\alpha}(u(t))+u(t)D_t^{\alpha}((v(t)))$. ii)

f) $D_t^{\alpha}\left(\frac{u(t)}{v(t)}\right)=\frac{v(t)D_t^{\alpha}(u(t))-u(t)D_t^{\alpha}((v(t))}{v^2(t)}, v(t)\neq 0.$ iii)

g) If, in addition to $u(t)$ differentiable, then $D_t^{\alpha}u(t)=t^{1-\alpha}\frac{du}{dt}$.

Theorem (ii): Let $\alpha\in(0,1)$ such $u(t)$ is differentiable and also $\alpha-$differentiable

Let $v(t)$ be a function defined in the range of $u(t)$ also differentiable, then

$$D_t^{\alpha}(u(t)\circ v(t))=t^{1-\alpha}v'(t)(u'(v(t)).$$

See also [16].

III. An Analysis of the Method

In this section, we present modified $tanh$ expansion method. For doing this we consider the following non-linear fractional differential equation in two independent variables *x* and *t*.

$$Q\left(u,D_t^{\alpha}u,D_x^{\alpha}u,D_{tt}^{2\alpha}u,D_{xx}^{2\alpha}u,D_t^{\alpha}D_t^{\alpha}u,..........\right)=0;0<\alpha<1 \quad (3)$$

where α is order of the derivative of the function $u=u(x,t)$. Also, we use the travelling wave variables,

$$u(x,t)=U(\xi)\text{and } \xi=Ax+B\frac{t^{\alpha}}{\alpha}, \quad (4)$$

where A and B are non-zero constants. After transformation we get the following ordinary differential equation (ODE):

$$Q(U,U',U'',..........)=0 \quad (5)$$

Where, ′ is a derivative with respect to ξ.

Suppose the solution of equation (5) can be expressed as follows:

$$u(x,t)=U(\xi)=a_0+\sum_{i=1}^{m}\left(a_i\Phi^i(\xi)+\frac{b_i}{\Phi^i(\xi)}\right) \quad (6)$$

where $a_0,a_i,b_i,(i=1,2,......m)$ are constants to be determined; m is a positive integer that non-linear terms in the equation and $\Phi(\xi)$ be satisfy the Riccati differential equation:

$$\Phi'(\xi)=\lambda+\Phi^2(\xi) \quad (7)$$

where λ is a constant. To form the general solutions of the Riccati Eq. (7), we select the following cases:

When $\lambda<0$, the general hyperbolic solutions of (7) are:

$$\Phi(\xi)=-\sqrt{-\lambda}\tanh(\sqrt{-\lambda}\xi)$$
$$\Phi(\xi)=-\sqrt{-\lambda}\coth(\sqrt{-\lambda}\xi).$$

When $\lambda>0$, the general trigonometric solutions of (7) are:

$$\Phi(\xi)=\sqrt{\lambda}\tan(\sqrt{\lambda}\xi)$$
$$\Phi(\xi)=-\sqrt{\lambda}\cot(\sqrt{\lambda}\xi).$$

When $\lambda=0$, then the normal solution is:

$$\Phi(\xi)=-\frac{1}{\xi}.$$

Thus putting equation (6) and its necessary derivatives into equation (5) gives a polynomial in $\Phi(\xi)$, collect all terms with the same power of $\Phi(\xi)$ together. Equating the coefficients of the polynomial to zero, we will get a set of over-determine algebraic equations for $a_0,a_i,b_i,(i=1,2,......m)$, and λ with the help of symbolic computation using Maple.

Finally, solving the algebraic equations and above possible solutions of Riccati equation into (5), we obtain the solution of equation (3).

IV. Apllication

We consider the conformable time fractional Hirota-Satsuma equations version of (1) the form

$$\begin{cases} D_t^{\alpha}u + u_{xxx} + 6uu_x - 6vv_x = 0, \\ D_t^{\alpha}v - 2v_{xxx} - 6uvu_x = 0. \end{cases} \quad (8)$$

Suppose that $u(x,t) = U(\xi)$, and

$$\xi = Ax + B\frac{t^{\alpha}}{\alpha}, \quad (9)$$

The travelling wave variable (9) permits one converting equation (8) into ODEs for $u = U(\xi)$ and $v = V(\xi)$ as follows:

$$\begin{cases} BU' + U''' + 6UU' - 6AVV' = 0 \\ BV' - 2A^3V''' - 6AUV' = 0 \end{cases} \quad (10)$$

Considering the homogeneous balancing between U''' and VV' and between V''' and UV' in equation (10) $m_1 + 3 = 2m_2 + 1$ and $m_2 + 3 = m_1 + m_2 + 1$ respectively.

So $m_1 = 2,\ m_2 = 2.$

Therefore, equation (8) has a solution of the form:

$$\begin{cases} U(\xi) = a_0 + a_1\Phi\ (\xi) + a_2\Phi^2(\xi) + \dfrac{b_1}{\Phi(\xi)} + \dfrac{b_2}{\Phi^2(\xi)} \\ V(\xi) = c_0 + c_1\Phi\ (\xi) + c_2\Phi^2(\xi) + \dfrac{d_1}{\Phi(\xi)} + \dfrac{d_2}{\Phi^2(\xi)} \end{cases} \quad (11)$$

where from equation (7)

$\Phi'(\xi) = \lambda + \Phi^2(\xi)$ and

$$\Phi''(\xi) = 2\left(\Phi(\xi)\lambda + \Phi^2(\xi)\right) \quad (12)$$

Substituting equation (11) and equation (12) into equation (8) yields a set of algebraic equations for $a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, d_1, d_2, A, B, \lambda.$ These algebraic equations system are obtained as

$$-48A^3c_2 - 12Aa_2c_2 = 0$$
$$-6Aa_2c_1 - 12A^3c_1 - 12Aa_1c_2 = 0$$
$$-80A^3c_2\lambda - 12\lambda Aa_2c_2 - 6Aa_1c_1 - 12Aa_0c_2 + 2Bc_2 = 0$$
$$-6\lambda Aa_2c_1 - 16A^3c_1\lambda - 12\lambda Aa_1c_2 - 12Ab_1c_2 + 6Aa_2d_1 + Bc_1 - 6Aa_0c_1 = 0$$
$$-32A^3c_2\lambda^2 - 6\lambda Aa_1c_1 - 12\lambda Aa_0c_2 + 2Bc_2\lambda - 6Ab_1c_1 + 12Aa_2d_2 + 6Aa_1d_1 - 12Ab_2c_2 = 0$$
$$-12\lambda Ab_1c_2 + 6\lambda Aa_2d_1 - 6\lambda Aa_0c_1 + 6Aa_0d_1 - 6Ab_2c_1 + 12Aa_1d_2 + Bc_1\lambda + 4A^3d_1\lambda - 4A^3c_1\lambda^2 - Bd_1 = 0$$
$$-6\lambda Ab_1c_1 + 12\lambda Aa_2d_2 + 6\lambda Aa_1d_1 - 12\lambda Ab_2c_2 - 2Bd_2 + 6Ab_1d_1 + 32A^3d_2\lambda + 12Aa_0d_2 = 0$$
$$6\lambda Aa_0d_1 - 6\lambda Ab_2c_1 + 16A^3d_1\lambda^2 + 12\lambda Aa_1d_2 - Bd_1\lambda + 6Ab_2d_1 + 12Ab_1d_2 = 0$$
$$-2Bd_2\lambda + 6\lambda Ab_1d_1 + 80A^3d_2\lambda^2 + 12\lambda Aa_0d_2 + 12Ab_2d_2 = 0$$
$$6\lambda Ab_2d_1 + 12\lambda Ab_1d_2 + 12A^3d_1\lambda^3 = 0$$
$$48A^3d_2\lambda^3 + 12\lambda Ad_2b_2 = 0$$

By using Maple solving the above system, we obtain

Set 1

$$a_0 = -\frac{\left(16\lambda A^3 - B\right)}{6A}, a_1 = 0, a_2 = -4A^2, b_1 = 0, b_2 = -4A^2\lambda^2, c_0 = \pm\frac{\left(-8\lambda A + 16\lambda A^3 - B - BA\right)\sqrt{4A^3 - 2A}}{6A\left(2A^2 - 1\right)},$$
$$c_1 = 0, c_2 = \pm 2\sqrt{4A^3 - 2A}, d_1 = 0, d_2 = \pm 2\lambda^2\sqrt{4A^3 - 2A}.$$

Set 2

$$a_0 = -\frac{\left(16\lambda A^3 - B\right)}{6A}, a_1 = 0, a_2 = 0,$$
$$b_1 = 0, b_2 = -4A^2\lambda^2, c_0 = \pm\frac{\left(-8\lambda A + 16\lambda A^3 - B - BA\right)}{3\sqrt{4A^3 - 2A}},$$
$$c_1 = 0, c_2 = 0, d_1 = 0, d_2 = \pm 2\lambda^2\sqrt{4A^3 - 2A}.$$

Hence the solution is:

Case-I When $\lambda < 0$, we get the following hyperbolic function solutions:

Family 1

$$u_1(x,t) = -\frac{\left(16\lambda A^3 - B\right)}{6A} + 4A^2\lambda\tanh^2\sqrt{-\lambda\xi} + \frac{4A^2\lambda}{\tanh^2\sqrt{-\lambda\xi}}. \quad (13)$$

$$u_2(x,t) = -\frac{\left(16\lambda A^3 - B\right)}{6A} + 4A^2\lambda\coth^2\sqrt{-\lambda\xi} + \frac{4A^2\lambda}{\coth^2\sqrt{-\lambda\xi}}. \quad (14)$$

Family 2

$$u_3(x,t) = -\frac{\left(16\lambda A^3 - B\right)}{6A} + \frac{4A^2\lambda}{\tanh^2\sqrt{-\lambda\xi}}. \quad (15)$$

$$u_4(x,t) = -\frac{\left(16\lambda A^3 - B\right)}{6A} + \frac{4A^2\lambda}{\coth^2\sqrt{-\lambda\xi}}. \quad (16)$$

Family 3

$$v_1(x,t) = \frac{\left(-8\lambda A + 16\lambda A^3 - B - BA\right)\sqrt{4A^3 - 2A}}{6A\left(2A^2 - 1\right)} - 2\left(\sqrt{4A^3 - 2A}\right)\lambda\tanh^2\sqrt{-\lambda\xi} - \frac{2\lambda\sqrt{4A^3 - 2A}}{\tanh^2\sqrt{-\lambda\xi}}. \quad (17)$$

$$v_2(x,t) = \frac{\left(-8\lambda A + 16\lambda A^3 - B - BA\right)\sqrt{4A^3 - 2A}}{6A\left(2A^2 - 1\right)} - 2\left(\sqrt{4A^3 - 2A}\right)\lambda\coth^2\sqrt{-\lambda\xi} - \frac{2\lambda\sqrt{4A^3 - 2A}}{\coth^2\sqrt{-\lambda\xi}}. \quad (18)$$

Family 4

$$v_3(x,t) = \frac{\left(-8\lambda A + 16\lambda A^3 - B - BA\right)}{3A\left(2A^2 - 1\right)} + \frac{2\lambda\sqrt{4A^3 - 2A}}{\tanh^2\sqrt{-\lambda\xi}}. \quad (19)$$

$$v_4(x,t) = \frac{\left(-8\lambda A + 16\lambda A^3 - B - BA\right)}{3A\left(2A^2 - 1\right)} + \frac{2\lambda\sqrt{4A^3 - 2A}}{\coth^2\sqrt{-\lambda\xi}}. \quad (20)$$

Case-II When $\lambda > 0$, we get the following trigonometric function solutions:

Family 5

$$u_5(x,t) = -\frac{(16\lambda A^3 - B)}{6A} - 4A^2\lambda\tan^2\sqrt{\lambda}\xi - \frac{4A^2\lambda}{\tan^2\sqrt{\lambda}\xi}. \quad (21)$$

$$u_6(x,t) = -\frac{(16\lambda A^3 - B)}{6A} - 4A^2\lambda\cot^2\sqrt{\lambda}\xi - \frac{4A^2\lambda}{\cot^2\sqrt{\lambda}\xi}. \quad (22)$$

Family 6

$$u_7(x,t) = -\frac{(16\lambda A^3 - B)}{6A} - \frac{4A^2\lambda}{\tan^2\sqrt{\lambda}\xi}. \quad (23)$$

$$u_8(x,t) = -\frac{(16\lambda A^3 - B)}{6A} - \frac{4A^2\lambda}{\cot^2\sqrt{\lambda}\xi}. \quad (24)$$

Family 7

$$v_5(x,t) = \frac{(-8\lambda A + 16\lambda A^3 - B - BA)\sqrt{4A^3 - 2A}}{6A(2A^2 - 1)} + 2\left(\sqrt{4A^3 - 2A}\right)\lambda\tan^2\sqrt{\lambda}\xi + \frac{2\lambda\sqrt{4A^3 - 2A}}{\tan^2\sqrt{\lambda}\xi}. \quad (25)$$

$$v_6(x,t) = \frac{(-8\lambda A + 16\lambda A^3 - B - BA)\sqrt{4A^3 - 2A}}{6A(2A^2 - 1)} + 2\left(\sqrt{4A^3 - 2A}\right)\lambda\cot^2\sqrt{\lambda}\xi + \frac{2\lambda\sqrt{4A^3 - 2A}}{\cot^2\sqrt{\lambda}\xi}. \quad (26)$$

Family 8

$$v_7(x,t) = \frac{(-8\lambda A + 16\lambda A^3 - B - BA)}{3A(2A^2 - 1)} - \frac{2\lambda\sqrt{4A^3 - 2A}}{\tan^2\sqrt{\lambda}\xi}. \quad (27)$$

$$v_8(x,t) = \frac{(-8\lambda A + 16\lambda A^3 - B - BA)}{3A(2A^2 - 1)} - \frac{2\lambda\sqrt{4A^3 - 2A}}{\cot^2\sqrt{\lambda}\xi}. \quad (28)$$

Case-III When $\lambda = 0$, we get the following rational function solutions:

Family 9

$$u_9(x,t) = -\frac{(16\lambda A^3 - B)}{6A} - \frac{4A^2}{\xi^2} - 4A^2\lambda^2\xi^2. \quad (29)$$

$$v_9(x,t) = \frac{(-8\lambda A + 16\lambda A^3 - B - BA)\sqrt{4A^3 - 2A}}{6A(2A^2 - 1)} + \frac{2\left(\sqrt{4A^3 - 2A}\right)}{\xi^2} + 2\lambda^2\left(\sqrt{4A^3 - 2A}\right)\xi^2 \quad (30)$$

Here the value of ξ in the above is given by

$$\xi = Ax + B\frac{t^\alpha}{\alpha}. \quad (31)$$

V. Graphical Representations of some of the Obtained Solutions

Now we give some graphical representations of the conformable time-fractional on Hirota-Satsuma equations at different time levels through different values of α: $u(x,t)$ is given figure-1 while their equivalent results of $v(x,t)$ are given in figure-2 respectively.

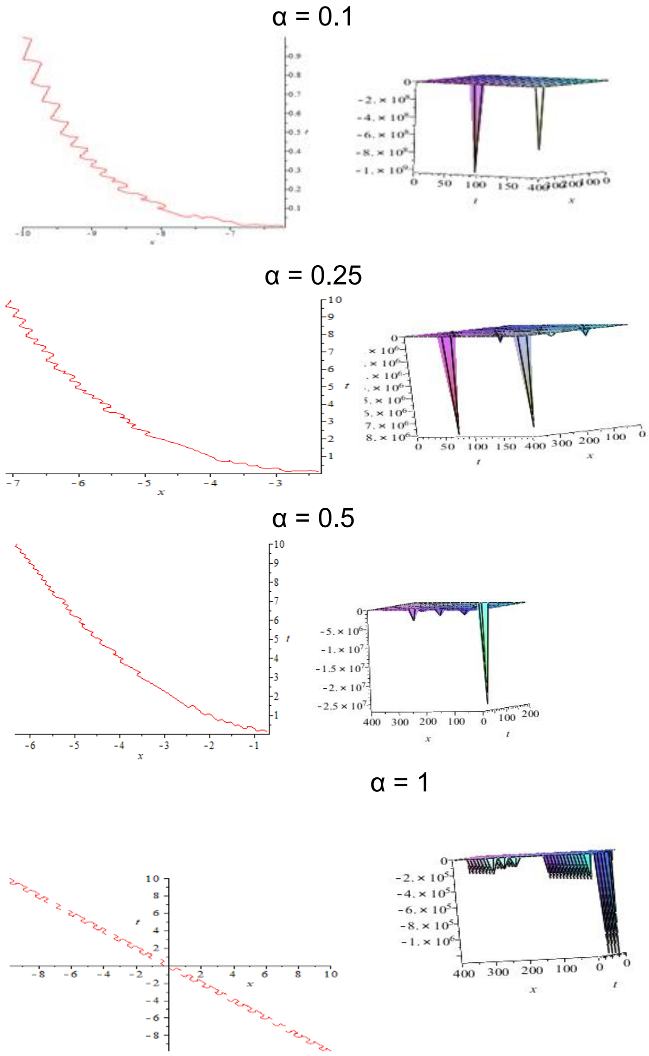
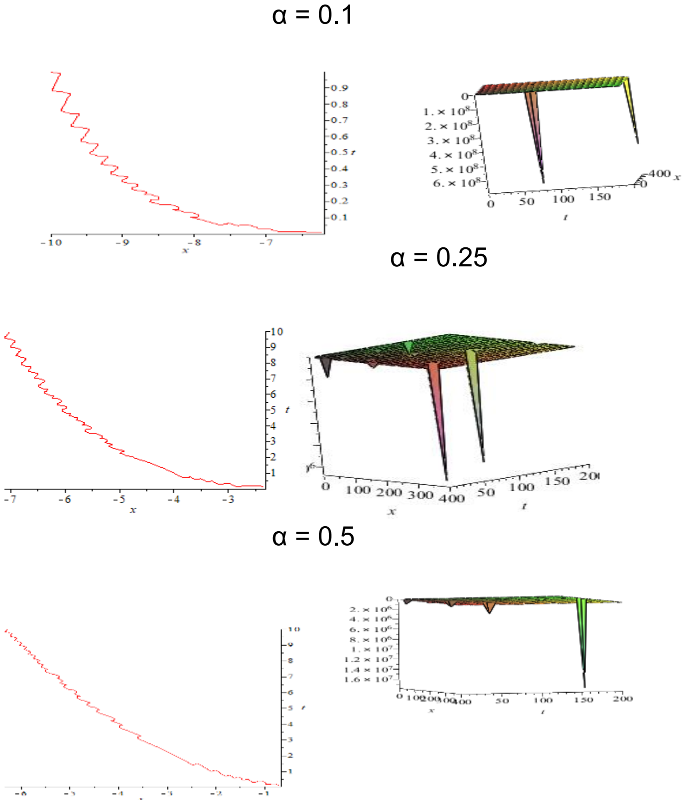


Figure-1 Profiles of $u(x,t)$ at different time levels



α = 1

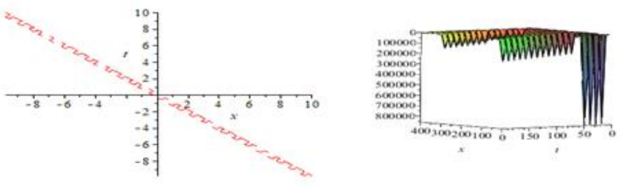


Figure-2 Profiles of $v(x,t)$ at different time levels

VI. CONCLUSION

In this work, various hyperbolic function solutions for the time fractional on Hirota-Satsuma equations are constructed using the $tanh$ expansion method in the sense of the newly devised conformable fractional derivative. We have found for Hirota-Satsuma equations the new families of solitary wave solutions and periodic solutions. The Maple software has been used for the solution of the system of algebraic equations and also for the graphical illustrations, respectively. We observed that the results of the time fractional on Hirota-Satsuma equations separate and produce a wave configuration when α is close to 0 (full memory), and the configuration vanishes when α is close to1.

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