Another Derivation Method Of The Formula Of Universal Gravitation

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***Abstract*— The law of universal gravitation is indirectly derived from Kepler's law and the equation of motion. The attraction between the planet and the Sun will be derived intricately (the concept of action-reaction law is essential) based on the state of the planet rotating around the Sun. Originally, it must be derived simply from the stationary state of the stationary planet and the Sun, that is, two objects. This paper directly derives the law of universal gravitational force acting between two stationary objects. Since the object does not need to rotate, the universal gravitational law formula can be derived regardless of the circular orbit or elliptical orbit.**

Keywords—Law of universal gravitation; gravity; General Relativity; space-time; continuum mechanics; acceleration field; curved space.

1. INTRODUCTION

Universal gravitation, or gravity, is everyone interested in the essence of the phenomenon. Why do apples fall? Why is there an attractive force between two objects? Fortunately, the formula of universal gravitational force was derived by Isaac Newton in 1665, and is phenomenologically used to explain observed facts and widely used in astronomical mechanics and spacecraft orbit calculations.
However, it is still unknown why and how gravity is generated.

The author tried to explain the cause of gravity by applying the mechanical structure of space to General Relativity, and by applying continuum mechanics to space. The apples on the Earth will not be pulled by the Earth and fall, but will be pushed and fall in the direction of the Earth due to the pressure of the field in the curved space area around the Earth.

Given a priori assumption that space as a vacuum has a physical fine structure like continuum, it enables us to apply a continuum mechanics to the so-called “vacuum” of space [1]. Minami proposed a hypothesis for mechanical property of space-time in 1988 [2]. A primary motive was to research in the realm of space propulsion theory using the substantial physical structure of space-time based on this hypothesis [2, 3].

He proposed that "gravity is a pushing force, not a pulling force" [4, 5, 6, 7]. In the book entitled “Mechanism of GRAVITY Generation —why apples fall—”[7], this book applies continuum mechanics and General Relativity to space, and explains the mechanism of gravity from a mechanical viewpoint of space in an easy-to-understand manner using diagrams. Although the mechanism of gravity is described in this book, in the appendix, the derivation of the formula for the attractive force between two directly stationary objects is shown instead of the derivation based on the conventional Kepler's law.

This paper introduces the direct derivation of the universal gravitational force equation between two stationary objects.

The following Chapter 2 explains the conventional derivation method of the formula of Universal Gravitation and Chapter 3 introduces another derivation method of the formula of Universal Gravitation.

2. Conventional Derivation Method of the Formula of Universal Gravitation

Newton's law of universal gravitation is usually stated as that every objects attracts every other object in the universe with a force, that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.

As is well known, the equation for universal gravitation thus takes the form:

$$F = G\frac{m_1 m_2}{r^2} \tag{2.1}$$

where F is the gravitational force acting between two objects, m_1 and m_2 are the masses of the objects, r is the distance between the centers of their masses, and G is the gravitational constant.

By applying the equation of motion to Kepler's law, the law of universal gravitation (law of inverse square) is derived. The following is a well-known derivation method, but it is described for reference. The magnitude of the attractive force acting between the planet and the Sun is derived as follows (see Fig. 2). To simplify the calculation, we assume that the planet is moving in a circular motion rather than an elliptical motion. The calculation for the elliptical orbit is more complicated than that for the circular orbit. Then, according to Kepler's second law, this circular motion is a uniform circular motion: this attractive force is the centripetal force acting on a planet that moves circularly.

The centripetal force F can be used as:

$$F = mr\omega^2 \tag{2.2}$$

where F is centripetal force, m is the mass of the planet, r is the radius of the circular orbit (distance between the planet and the Sun), ω is the angular velocity, T is the rotation period.

Substituting $\omega = \frac{2\pi}{T}$ into Eq.(2.2), we get

$$F = mr\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 mr}{T^2} \tag{2.3}$$

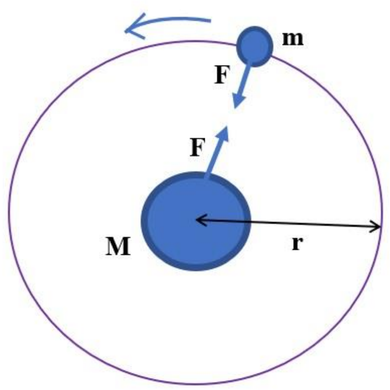

Fig. 2. Centripetal force F between objects M & m.

Next, from Kepler's third law, we have

$$\frac{T^2}{r^3} = k \tag{2.4}$$

Substituting Eq.(2.4) into Eq.(2.3), thus, we get

$$F = \frac{4\pi^2 m}{kr^2} = G\frac{m}{r^2} \qquad \left(\frac{4\pi^2}{k} \Rightarrow G\right) \tag{2.5}$$

Eq.(2.5) means that the magnitude of the attractive force F acting on the planet is proportional to the mass of the planet and inversely proportional to the square of the distance from the Sun.

By the way, from the relation of action and reaction, the opposite direction and the same force F should act on the Sun. From the symmetry with F, this force is considered to be proportional to the mass M of the Sun and inversely proportional to the square of r.

Eq.(2.5) obtained above is an attractive force for the planet, but according to the law of action-reaction, the same magnitude of force F should be exerted on the Sun.
The magnitude of the attractive force acting on the Sun is also proportional to the mass of the Sun and inversely proportional to the square of the distance from the planet.
To satisfy above both states at the same time, the attractive force acting between the Sun and the planet is proportional to the mass of the Sun (M), proportional to the mass of the planet (m), and inversely proportional to the square of the distance between the sun and the planet.
Must be if this is expressed by an equation (the mass of the sun is M and the constant is G),

$$F = G\frac{Mm}{r^2} \tag{2.6}$$

In this way, the conventional derivation method uses centripetal force while the object is rotating, so derivation of the attractive force between two objects is indirect.
In addition, since the attraction between two objects is mentioned, the law of action-reaction will be brought up and explained.
What makes me more suspicious than anything else is that the attraction between the two objects must be directly derived while the two objects are originally stationary.

3. Another Derivation Method of the Formula of Universal Gravitation

3. 1 Derivation of the Formula of Universal Gravitation

A massive body causes the curvature of space-time around it, and a free particle responds by moving along a geodesic line in that space-time. The path of free particle is a geodesic line in space-time and is given by the following geodesic equation;

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\cdot\frac{dx^j}{d\tau}\cdot\frac{dx^k}{d\tau} = 0 \tag{3.1}$$

where $\Gamma^i{}_{jk}$ is Riemannian connection coefficient, τ is proper time, x^i is four-dimensional Riemann space, that is, three dimensional space ($x=x^1$, $y=x^2$, $z=x^3$) and one dimensional time ($w=ct=x^0$), where c is the velocity of light. These four coordinate axes are denoted as x^i (i=0, 1, 2, 3).

Proper time is the time to be measured in a clock resting for a coordinate system. We have the following relation derived from an invariant line element ds^2 between Special Relativity (flat space) and General Relativity (curved space):

$$d\tau = \sqrt{-g_{00}}dx^0 = \sqrt{-g_{00}}cdt \quad (3.2)$$

From Eq.(3.1), the acceleration of free particle is obtained by

$$\alpha^i = \frac{d^2x^i}{d\tau^2} = -\Gamma^i_{jk}\cdot\frac{dx^j}{d\tau}\cdot\frac{dx^k}{d\tau} \quad (3.3)$$

As is well known in General Relativity, in the curved space region, the massive body "*m* (kg)" existing in the acceleration field is subjected to the following force F^i (N) :

$$F^i = m\Gamma^i_{jk}\cdot\frac{dx^j}{d\tau}\cdot\frac{dx^k}{d\tau} = m\sqrt{-g_{00}}c^2\Gamma^i_{jk}u^ju^k = m\alpha^i \quad (3.4)$$

where u^j, u^k are the four velocity, Γ^i_{jk} is the Riemannian connection coefficient, and τ is the proper time.
From Eqs.(3.3),(3.4), we obtain:

$$\alpha^i = \frac{d^2x^i}{d\tau^2} = -\Gamma^i_{jk}\cdot\frac{dx^j}{d\tau}\cdot\frac{dx^k}{d\tau} = -\sqrt{-g_{00}}c^2\Gamma^i{}_{jk}u^ju^k \quad (3.5)$$

Eq.(3.5) yields a more simple equation from the condition of linear approximation, that is, weak-field, quasi-static, and slow motion (speed v << speed of light *c*: $u^0 \approx 1$):

$$\alpha^i = -\sqrt{-g_{00}}\cdot c^2\Gamma^i_{00} \quad (3.6)$$

On the other hand, the major component of spatial curvature R^{00} in the weak field is given by

$$R^{00} \approx R_{00} = R^{\mu}_{0\mu 0} = \partial_0\Gamma^{\mu}_{0\mu} - \partial_{\mu}\Gamma^{\mu}_{00} + \Gamma^{\nu}_{0\mu}\Gamma^{\mu}_{\nu 0} - \Gamma^{\nu}_{00}\Gamma^{\mu}_{\nu\mu} \quad (3.7)$$

In the nearly Cartesian coordinate system, the value of $\Gamma^{\mu}_{\nu\rho}$ are small, so we can neglect the last two terms in Eq.(3.7), and using the quasi-static condition we get

$$R^{00} = -\partial_{\mu}\Gamma^{\mu}_{00} = -\partial_i\Gamma^i_{00} \quad (3.8)$$

From Eq.(3.8), we get formally

$$\Gamma^i_{00} = -\int R^{00}(x^i)dx^i \quad (3.9)$$

Substituting Eq.(3.9) into Eq.(3.6), we obtain

$$\alpha^i = \sqrt{-g_{00}}c^2\int R^{00}(x^i)dx^i \quad (3.10)$$

Accordingly, from the following linear approximation scheme for the gravitational field equation:(1) weak gravitational field, i.e. small curvature limit, (2) quasi-static, (3) slow-motion approximation (i.e., $v/c << 1$), and considering range of curved region, we get the following relation between acceleration of curved space and curvature of space:

$$\alpha^i = \sqrt{-g_{00}}c^2\int_a^b R^{00}(x^i)dx^i \quad (3.11)$$

where α^i : acceleration (m/s²), g_{00}: time component of metric tensor, a-b: range of curved space (m), x^i: components of coordinate (*i*=0,1,2,3), *c*: velocity of light, R^{00} : major component of spatial curvature(1/m²).

Eq.(3.11) indicates that the acceleration field α^i is produced in curved space. The intensity of acceleration produced in curved space is proportional to the product of spatial curvature R^{00} and the length of curved region.

Eq.(3.4) yields more simple equation from above-stated linear approximation ($u^0 \approx 1$),

$$F^i = m\sqrt{-g_{00}}c^2\Gamma^i_{00}u^0u^0 = m\sqrt{-g_{00}}c^2\Gamma^i_{00} = m\alpha^i$$
$$= m\sqrt{-g_{00}}c^2\int_a^b R^{00}(x^i)dx^i \quad (3.12)$$

Setting *i*=3 (i.e., direction of radius of curvature: *r*), we get Newton's second law:

$$F^3 = F = m\alpha = m\sqrt{-g_{00}}c^2\int_a^b R^{00}(r)dr = m\sqrt{-g_{00}}c^2\Gamma^3_{00} \quad (3.13)$$

The acceleration (α) of curved space and its Riemannian connection coefficient (Γ^3_{00}) are given by:

$$\alpha = \sqrt{-g_{00}}c^2\Gamma^3_{00}\ , \quad \Gamma^3_{00} = \frac{-g_{00,3}}{2g_{33}} \quad (3.14)$$

where *c*: velocity of light, g_{00} and g_{33}: component of metric tensor, $g_{00,3}$: $\partial g_{00}/\partial x^3 = \partial g_{00}/\partial r$. We choose the spherical coordinates "$ct=x^0$, $r=x^3$, $\theta=x^1$, $\varphi=x^2$ " in space-time. The acceleration α is represented by the equation both in the differential form and in the integral form. Practically, since the metric is usually given by the solution of gravitational field equation, the differential form has been found to be advantageous.

Now in general, the line element is described in:

$$ds^2 = g_{ij}dx^idx^j = g_{00}(dx^0)^2 + g_{33}(dx^3)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2$$
$$= g_{00}(cdt)^2 + g_{33}(dr)^2 + g_{11}r^2(d\theta)^2 + g_{22}r^2\sin^2\theta(d\varphi)^2 \quad (3.15)$$

We choose the spherical coordinates "ct=x^0, r=x^3,θ=x^1, φ=x^2 " in space-time (see Fig. 3.1).

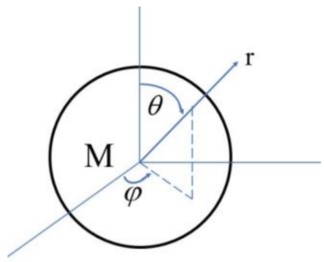

Fig. 3.1. Spherical coordinates.

Next, let us consider External Schwarzschild Solution.

External Schwarzschild Solution is an exact solution of the gravitational field equation, which describes the gravitational field outside the spherically symmetric, static mass distribution.

The line element is obtained as follows:

$$ds^2 = -(1-\frac{r_g}{r})c^2dt^2 + \frac{1}{1-\frac{r_g}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{3.16}$$

The metrics are given by:

$$g_{00} = -(1-r_g/r), g_{11} = g_{22} = 1, g_{33} = 1/(1-r_g/r), \quad and\ other\ g_{ij} = 0\ . \tag{3.17}$$

where r_g is the gravitational radius (i.e. $r_g = 2GM/c^2$).

Combining Eq.(3.17) with Eq.(3.14) yields:

$$\alpha = G \cdot \frac{M}{r^2}, (r_g \langle\ r) \tag{3.18}$$

where G is a gravitational constant and M is a total mass.

Eq.(3.18) indicates the acceleration at a distance "r" from the center of the well-known the Earth mass M.
The force acting on a mass "m" located at a distance "r" from the center of the Earth mass M is:

$$F = m\alpha = mG\frac{M}{r^2} = G\frac{Mm}{r^2} \tag{3.19}$$

Eq.(3.19) indicates a universal gravitational force acting on masses M and m that are stationary with respect to each other.

<Supplemental explanation for Eq.(3.18)>

$$\alpha = \sqrt{-g_{00}}c^2\Gamma^3_{00}\ , \quad \Gamma^3_{00} = \frac{-g_{00,3}}{2g_{33}} \tag{3.14}$$

The metrics are given by (Eq.3.17)

$$g_{00} = -\left(1-\frac{r_g}{r}\right) = -1+\frac{r_g}{r}\ , \quad g_{33} = \frac{1}{1-\frac{r_g}{r}}$$

Using Eq.(3.14),

$$g_{00,3} = \frac{\partial g_{00}}{\partial x^3} = \frac{\partial g_{00}}{\partial r} = \frac{\partial}{\partial r}\left(-1+\frac{r_g}{r}\right) = -\frac{r_g}{r^2}$$

$$\Gamma^3_{00} = \frac{-g_{00,3}}{2g_{33}} = \frac{r_g}{r^2} \cdot \frac{1-\frac{r_g}{r}}{2} = \frac{r_g}{2r^2} - \frac{r_g{}^2}{2r^3} \approx \frac{r_g}{2r^2}$$

Since $r_g = \frac{2GM}{c^2}$ is the gravitational radius, then $r_g \ll r$. Accordingly, the term of $\frac{r_g{}^2}{2r^3}$ is neglected.

$$g_{00} = -\left(1-\frac{r_g}{r}\right) \approx -1\ .$$

Acceleration (α) is obtained (3.18) :

$$\alpha = \sqrt{-g_{00}}c^2\Gamma^3_{00} = c^2\Gamma^3_{00} = c^2\frac{r_g}{2r^2} = c^2\frac{1}{2r^2}\frac{2GM}{c^2} = \frac{GM}{r^2}$$

3. 2 Brief Summary of Mechanism of GRAVITY

As shown in Fig. 3.2, the gravitational field around the Earth is multiply covered by concentric or spherical curved spaces centered on the Earth.

Considering the case of the Earth, the curvature of space is spherically symmetric about the Earth and is fixed to the Earth, so the Earth itself cannot move due to the curvature of the space generated by the Earth.
However, as shown in Fig. 3.3, the apples on the Earth are independently in the curved spatial region of the Earth. Since the apple exists in the curved spatial region from the curved spatial layer at the apple's position to the curved spatial layer at the distant position, the apple is pushed by the generated curved space (i.e., pressure) and falls. That is, a sort of graduated pressure field is generated by the curved range from an arbitrary point "a" in curved space to a point "b" (the point at which space is absent of curvature, i.e., flat space of curvature "0") as shown in Fig. 3.4. Then apple moves directly towards the center of the Earth, that is, the apple falls. Falling acceleration of apple in curved space is proportional to both the value of spatial curvature and the size of curved space.

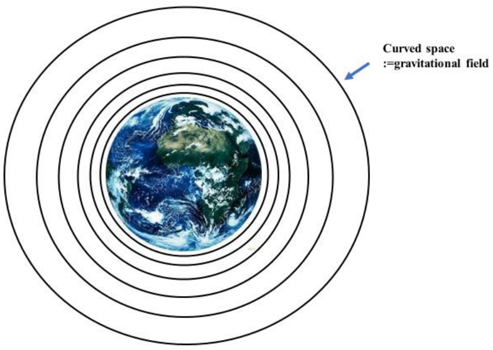

Fig. 3.2. Gravitational field around the Earth is a curved space that is concentric or spherical about the Earth.

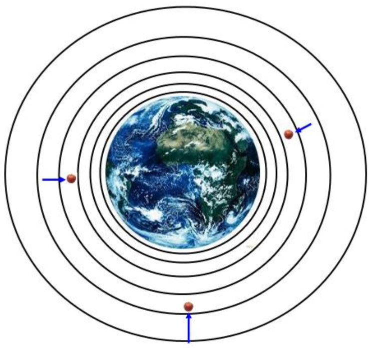

Fig. 3.3. Since the apples on the Earth are independently in the curved spatial regions of the Earth, the apples fall under the pressure generated in the curved spatial regions.

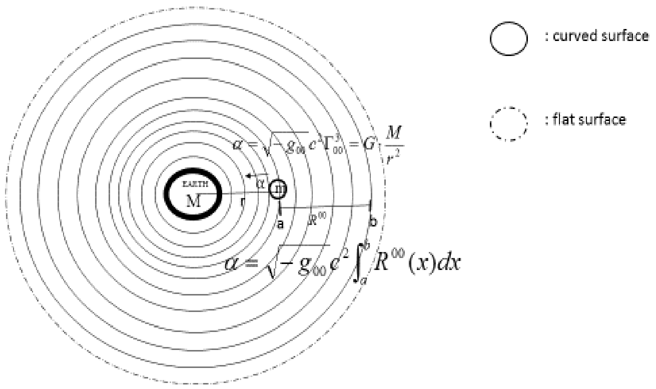

Fig. 3.4. Apple falls receiving a pressure of the field.

Next, consider the universal gravitational force of a well-known apple falling to the Earth.
Although the attraction between the Earth and the apple by universal gravitation can be explained by a mathematical formula, $F=G\dfrac{Mm}{r^2}$, there is no explanation of the mechanism of the attraction, that is, the principle of operation.
The mechanism can be understood by interpreting that the Earth and the apple are pushed toward each other from behind the curved space area around the Earth and the curved space area around the apple.
A phenomenon is that an apple is not pulled and falls by the Earth, but the apple is pushed toward the Earth under the pressure of the vast curved space area of the Earth.

Fig. 3.5 shows the mechanism.
In the upper diagram of Fig. 3.5, there are mass bodies A and B, and the space around each mass body is curved. As already explained, the mass B is pushed out of the curved space field generated by the mass A, and the mass A is pushed out of the curved space field generated by the mass B, so that they will move in the direction of opposition to each other.
In the lower diagram of Fig. 3.5, mass A is a giant mass of the Earth, and mass B is a light apple.

Apple is pushed from the vast curved space area of the Earth and go straight to the Earth. On the other hand, the Earth is also pushed from the narrow-curved space area of the apple and go straight to the apple. Since the mass of an apple is smaller than that of the Earth, the range of the curved space is small and the acceleration with respect to the Earth is almost zero.
In effect, it looks like an apple is pulled by the Earth and falls. Please refer to Ref. [7] in detail.

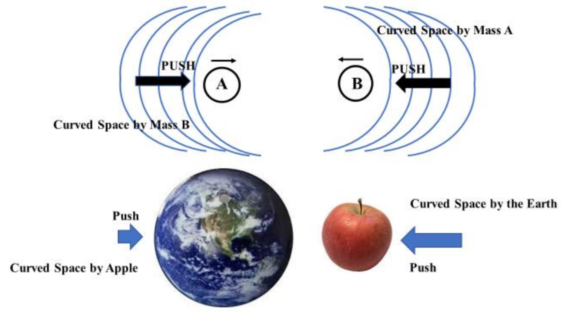

Fig. 3.5. Apple and the Earth are pushed out of a curved space and collide.

4. Conclusion

Assuming that space is an infinite continuum, a mechanical concept of space became identified. Space can be considered as a kind of transparent elastic field. The pressure field derived from the geometrical structure of space is newly obtained by applying both continuum mechanics and General Relativity to space.
The mass on the Earth will not be pulled by the Earth and fall, but will be pushed and fall in the direction of the Earth due to the pressure of the field in the curved space area around the Earth. Gravitation can be explained as a pressure field induced by the curvature of space.

Originally, the law of universal gravitation must be derived simply from the stationary state of two objects. This paper directly derives the law of universal gravitational force acting between two stationary objects. Since the object does not need to rotate, the universal gravitational law formula can be derived regardless of the circular orbit or elliptical orbit.
Instead of deriving based on Kepler's law, we could derive the universal gravitational force between two stationary objects directly from the field of a curved space.

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