Approximate Solution for Van Der Pol Equation By Adomian Decomposition Method

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Abstract: We will show Adomian's Decomposition Method (ADM) for solving Van der Pol equation, we will write the Van der Pol equation as the following forms:

$$y'' + y = g(x) + \mu(1 - y^2)y', \qquad (1)$$

and we will give new differential operators *L* for the left hand (1). We conclude that the resulted solutions using these differential operators are convergent. We also presented examples using (ADM) to explain this method. (ADM) is an effective and quick method. It contains easy calculations and simple operations.

Keywords: Adomian Decomposition Technique, Van der Pol equation, initial conditions.

1. Introduction

In this study, we introduce a new technique of Adomian decomposition method for solving Van der Pol's equation that gives the form:

$$y'' - \mu(1 - y^2)y' + y = g(x). \qquad (2)$$

Under the following initial conditions

$$y(0) = A, y'(0) = B,$$

where μ is secular parameter, g(x) is given function. The Van der Pol equation is significant in all different sciences, and is considered that finding an exact solution of a mathematical model as one of the most problematic topics in mathematics. Balthzar Van der Pol carried some experiments on dynamics in the laboratory during the (1920) and (1930)s. He first introduced equation in order to describe triode oscillations in electric circuits [11]. The Van der Pol equation with large value of non-linearity parameter has been studied by Cartwright and Littlewood in (1945) [5], they showed that the singular solution existed too. Moreover, the Van der Pol equation for non- linear plasma oscillations has been studied by Hafeez and Chifu in 2014 [7]. Many researchers have tried to solve and study the Van der pol equation in various forms. In (2001) Mickens proposed the study of a non-standard finite difference scheme. In (2002) Mickens studied numerically the Van der equation. In(2003) Mickens proposed different forms of fractional Van der Pol oscillators [10]. However, in recent studies, we can observe an explosive growth of methods for finding exact solutions of non-linear differential equations. To cite a few examples, here are some of the most popular of them: the tanh-function method [4], the Jacobi elliptic function expansion method [9], the Simplest equation method [12], the Exp-function method [8]. In (1970) George Adomian concoct a new method of solving non-linear differential equations [1]. The method was named Adomian decomposition method (AADM). This manner was used to get an exact solution of linear and non-linear equations. This method has been applied by many researchers in solving nonlinear equations [2,3,6,13]. The Adomian Decomposition Manner is used by separating the equation into linear and non-linear parts. Inverting the higher order derivative operator comprise in the linear operator on together sides. Using the initial or boundary conditions at first part of the series, and the nonlinear part is decomposition into a series of polynomials called Adomian polynomials. These manner detections approximate solution by takeover feature of the tiny parameter that show in the initial value question. Non-linear Oscillator equation has been used in many regions of physics and engineering. These systems are significant in mechanical and dynamics. By Adomian Decomposition Manner, we suggest a new differential operators and invers operators which can be used for solving Van der Pol equation.

2. Adomian decomposition method for Van der Pol Equation

We take the differential operators L in the part of $y'' + y$, then (1) we can rewrite in the following form:

$$Ly = g(x) + \mu(1 - y^2)y', \quad (3)$$

We suggest new differential operators L as below

$$L(.) = e^{-ix}\frac{d}{dx}e^{2ix}\frac{d}{dx}e^{-ix}(.), \quad (4)$$

$$L(.) = \frac{1}{cosx}\frac{d}{dx}cos^2x\frac{d}{dx}\frac{1}{cosx}(.), \quad (5)$$

$$L\,(.) = \frac{1}{sinx}\frac{d}{dx}sin^2x\frac{d}{dx}\;\frac{1}{sinx}(.), \quad (6)$$

The invers differential operators L^{-1} are defined respectively as

$$L^{-1}(.) = e^{ix}\int_0^x e^{-2ix}\int_0^x e^{ix}(.)\ dxdx\ , \qquad (7)$$

$$L^{-1}(.) = cosx\int_0^x cos^{-2}x\int_0^x cosx\,(.)dxdx\ , \qquad (8)$$

$$L^{-1}(.) = \ sin\ x\int_c^x sin^{-2}x\int_0^x sinx\,(.)dxdx\ . \qquad (9)$$

Operating with invers differential operators (7) and (8) on (3), under the initial conditions $y(0) = A, y'(0) = B$, we have

$$y(x) = \ Acosx + Bsinx + L^{-1}g(x) + \ \mu L^{-1}(1 - y^2)y'. \qquad (10)$$

also, Operating with invers differential operator (9) on (3), under the boundary conditions $y(c) = A, y'(0) = B$, we have

$$y(x) = sin\ x\left(\frac{1}{sinc}\right)A + \left(cos\ x - sin\ x\ cot\ (c)\right)B + L^{-1}g(x) + \ \mu L^{-1}(1 - y^2)y'. \qquad (11)$$

According to the ADM, the solution $y(x)$ is represented by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x)\ , \qquad (12)$$

and the non-linear term [14], for equation (1), shown by the decomposition series

$$N\big(y(x)\big) = \mu(1 - y_n^2)y_n' = \mu(1 - A_n)y_n'\ , \qquad (13)$$

where $A_n(x)$, the Adomian polynomials, defined by

$$A_n \ = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)]_{\lambda=0}\ , n = 0,1,2,\ldots$$

Substituting (12) and (13) in (10), we have

$$\sum_{n=0}^{\infty} y_n(x) = \ Acosx \ + \ Bsinx + L^{-1}(g(x) + \mu L^{-1}\sum_{n=0}^{\infty}(1 - A_n)y_n'. \qquad (14)$$

Can write recursive formula as

$$y_0(x) = \ Acosx + Bsinx \ + \ L^{-1}g(x), \qquad (15)$$

and

$$y_{n+1}(x) = \mu\, L^{-1}(1 - A_n)y_n', n \geq \ 0, \qquad (16)$$

and substituting (12) and (13) in (11), we have

$$\sum_{n=0}^{\infty} y_n(x) = sin\ x\left(\frac{1}{sinc}\right)A + \left(cos\ x - sin\ x\ cot(c)\right)B + \ L^{-1}\left(g(x)\right)$$

$$+ \mu\, L^{-1}\sum_{n=0}^{\infty}(1 - A_n)y_n'. \qquad (17)$$

Can write recursive formula as:

$$y_0(x) = sin\ x(\frac{1}{sinc})A + (cos\ x - sin\ x\ cot(c))B + \ L^{-1}(g(x),$$

$$y_{n+1}(x) = \mu L^{-1}\sum_{n=0}^{\infty}(1 - A_n)y_n' \ , n \geq \ 0\ ,$$

For numerical purposes, the n-part approximant

$$\emptyset_n(x) = \sum_{i=0}^{n-1} y_i\ ,$$

With

$$y(x) = \lim_{n\to\infty} \emptyset_n\ (x).$$

3. Application of ADM for Van der Pol’s equation

In this section, we will show some illustrative examples using three different differential operators to explain the Van der Pol equation by (ADM), as follows

Example 1.

Consider the Van der Pol’s equation:

$$y'' - \mu(1 - y^2)y' + y \ = \ 5e^{2x} + e^{4x} - y^2, \qquad (18)$$

$$y(0) = 1\ , y'\ (0) = 2,$$

with exact solution $y(x) = e^{2x}$ put $\mu = 0$.

In an operator form (4), equation (18) becomes

$$Ly = 5e^{2x} + e^{4x} - y^2, \quad (19)$$

using the invers differential operator L^{-1} on(19), we have

$$y(x) = L^{-1}(5e^{2x} + e^{4x}) - L^{-1}(y^2), \quad (20)$$

under the initial conditions $y(0) = 1,\ y'(0) = 2,$ the solution yields

$$y = cosx + 2sinx + L^{-1}(5e^{2x} + e^{4x}) - L^{-1}(y^2), \quad (21)$$

So

$$y_0 = cosx + 2sinx + L^{-1}(5e^{2x} + e^{4x}),$$

$$y_{n+1}(x) = -L^{-1}(y^2), n \geq 0, \quad (22)$$

The Adomian polynomials for the non-linear parts $f(y) = y^2,$, as follows

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

now, we will find the components $y_0, y_1, y_2,$ as follows

$$y_0 = 1 + 2x + \frac{5x^2}{2} + 2x^3 + \frac{37x^4}{24} + \frac{53x^5}{60} + \frac{67x^6}{144} + \frac{197x^7}{840} + \frac{631x^8}{5760} + \frac{8513x^9}{181440} + \frac{1909x^{10}}{103680} + \cdots,$$

$$y_1 = -\frac{x^2}{2} - \frac{2x^3}{3} - \frac{3x^4}{4} - \frac{7x^5}{10} - \frac{26x^6}{45} - \frac{269x^7}{630} - \frac{2911x^8}{10080} - \frac{16271x^9}{90720} - \frac{2959x^{10}}{28350} - \cdots,$$

$$y_2 = \frac{x^4}{12} + \frac{x^5}{6} + \frac{2x^6}{9} + \frac{73x^7}{315} + \frac{4289x^8}{20160} + \frac{15881x^9}{90720} + \frac{60079x^{10}}{3628800} + \ldots$$

Therefore

$$y(x) = y_0 + y_1 + y_2 =$$

$$1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{7x^4}{8} + \frac{7x^5}{20} + \frac{79x^6}{720} + \frac{11x^7}{280} + \frac{193x^8}{5760} + \frac{7733x^9}{18440} + \frac{169079x^{10}}{3628800} + \cdots,$$

Table 1. Difference between ADM and Exact solution

X	*Exact solution*	*ADM*	*Absolut Error*
0.0	*1.00*	*1.0000*	*0.0000*
0.1	*1.2214*	*1.22142*	*0.00002*
0.2	*1.49182*	*1.49219*	*0.00037*
0.3	*1.82212*	*1.82403*	*0.00191*
0.4	*2.22554*	*2.23187*	*0.00633*
0.5	*2.71828*	*2.73457*	*0.0162*

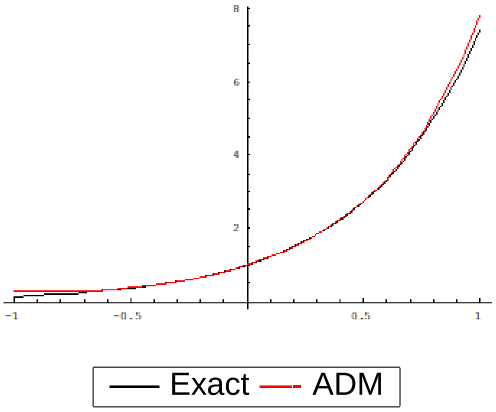

Figure 1: The Approximation solution for ADM and Exact solution

Figure 1. The Approximation for ADM and Exact solution $y = e^{2x}$
Table 1. The difference between the results we have got by (ADM) and exact solution.
Figure 1. We noted that the solutions are approximate, although we have found y_0, y_1, y_2, only.

Example 2.
Assume the Van der pol's equation

$$y'' - \mu(1-y^2)y' + y = 2 - 2x + x^2 + 2x^5, \tag{23}$$

rearrange equation (23), we get

$$y'' + y = 2 - 2x + x^2 + 2x^5 + \mu(1 - y^2)y', \quad (24)$$

we will put $\mu = 1$.

In an operator form (5), equation (24) becomes

$$Ly = 2 - 2x + x^2 + 2x^5 + (1 - y^2)y', \quad (25)$$

using invers differential operator L^{-1} on (25), under the initial conditions $y(0) = 0, y'(0) = 0$, the solution yields

$$y(x) = L^{-1}(2 - 2x + x^2 + 2x^5) + L^{-1}(1 - y^2)y' \quad (26)$$

where

$$y_0 = L^{-1}(2 - 2x + x^2 + 2x^5),$$

and

$$y_{n+1} = L^{-1}(1 - y_n^2)y_n', \quad (27)$$

where the polynomials Adomian for $f(y) = y^2$, are

$$A_0 = f(y_0) = y_0^2,$$

$$A_1 = y_1 f'(y_0) = 2y_0 y_1,$$

So

$$y_0 = x^2 - \frac{x^3}{3} + \frac{x^5}{60} + \frac{17x^7}{360} - \frac{17x^9}{25920},$$

$$y_1 = \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{180} - \frac{17x^7}{360} + \frac{319x^8}{6720} - \frac{101x^9}{8640} - \frac{31x^{10}}{30240} + \cdots,$$

$$y_2 = \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{180} + \frac{x^7}{840} - \frac{13x^8}{2240} - \frac{121x^9}{30240} + \frac{2579x^{10}}{453600} - \cdots$$

Thus

$$y(x) = y_0 + y_1 + y_2 =$$

$$x^2 - \frac{x^5}{60} + \frac{x^7}{840} + \frac{x^8}{24} - \frac{1483x^9}{90720} + \frac{151x^{10}}{x32400}.$$

Table 2. Difference between ADM and Exact solution $y = x^2$.

x	Exact solution	ADM	Absolute Error
0.0	0.00	0.0000	0.0000
0.1	0.01	0.00999983	0.00000017
0.2	0.04	0.0399948	0.0000052
0.3	0.09	0.0899622	0.0000378
0.4	0.16	0.159855	0.000145
0.5	0.25	0.249624	0.000376
0.6	0.36	0.359301	0.000699

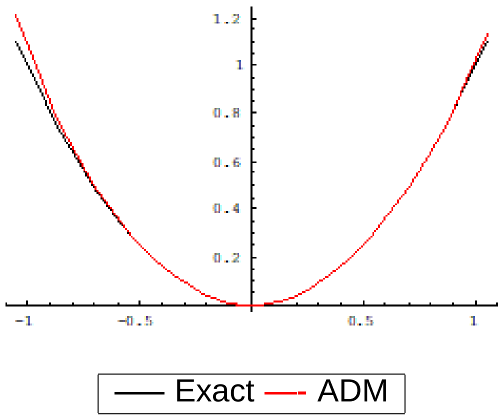

Figure 2: The Approximation solution for ADM and Exact solution.

Figure 2. The Approximation for ADM and Exact solution $y = x^2$

Table 2. The difference between the results we have got by (ADM) and exact solution.

Figure 2. We noted that the solutions are approximate, although we have found y_0, y_1, y_2, only.

Example 3.

Consider the Van der Pol's equation

$$y'' - \mu(1 - y^2)y' + y = 6x + x^3 + x^6 - y^2, \qquad (28)$$

$y(0) = 0$, y $(0.5) = 0.125$, and $\mu = 0$.

With exact solution $y(x) = x^3$,

In an operator form (6), equation (28) becomes:

$$Ly = 6x + x^3 + x^6 - y^2, \qquad (29)$$

using the invers differential operator L^{-1} on (29), and using boundary conditions $y(0) = 0$, $y(0.5) = 0.125$, we have

$$y(x) = \left(\frac{0.125}{sin\ (0.5)}\right) sin\ x + L^{-1}(6x + x^3 + x^6) - L^{-1}(y^2), \qquad (30)$$

$$y_0\ (x) = \left(\frac{0.125}{sin\ (0.5)}\right) sin\ x + L^{-1}(6x + x^3 + x^6),$$

$$y_{n+1}(x) = -L^{-1}(y^2), n \geq 0,$$

The Adomian polynomials for the non-linear parts $f(y) = y^2$, as follows

$$A_0 = y_0^2,$$
$$A_1 = 2y_0y_1,$$

In general, the first few components as below

$$y_0 = -0.000145093\,x + 1.00002\,x^3 - 1.2091 \times 10^{-6}x^5 - 2.22045 \times 10^{-16}x^6 + 2.87882 \times 10^{-8}x^7 + 0.0178571\,x^8 - 3.99836 \times 10^{-10}x^9 - 0.000198413x^{10}$$

$$y_1 = 0.000144751\,x - 1.61199 \times 10^{-19}\,x^2 - 0.0000241251x^3 - 1.75432 \times 10^{-9}\,x^4 + 1.20626 \times 10^{-6}x^5 + 9.67313 \times 10^{-6}x^6 - 2.87204 \times 10^{-8}x^7 - 0.0178582\,x^8 + 3.98895 \times 10^{-10}x^9 + 0.000198451\,x^{10},$$

$$y_2 = 3.05988 \times 10^{-7}x - 6.55512 \times 10^{-23}x^2 - 5.0998 \times 10^{-8}x^3 + 3.50038 \times 10^{-9}x^4 + 2.5499 \times 10^{-9}x^5 - 9.65064 \times 10^{-6}x^6 - 6.07241 \times 10^{-11}x^7 + 1.03398 \times 10^{-6}x^8 + 8.85618 \times 10^{-11}x^9 - 3.82958 \times 10^{-8}x^{10}.$$

So

$$y(x) = y_0 + y_1 + y_2 =$$

$$-3.56829 \times 10^{-8}x - 1.61264 \times 10^{-19}x^2 + 1.x3 + 1.7460610^{-9}x^4 - 2.97358 \times 10^{-10}x^5 + 2.24871 \times 10^{-8}x^6 + 7.06783 \times 10^{-12}x^7 - 2.41868 \times 10^{-9}x^8 + 8.7620210^{-11}x^9 + 8.94076 \times 10^{-11}x^{10}.$$

Table 3. Difference between

x	Exact solution	ADM	Absolute Error
0.1	0.001	0.0009999	0.0000001
0.2	0.008	0.0079999	0.0000001
0.3	0.027	0.027	0.0000
0.4	0.064	0.064	0.0000
0.5	0.125	0.125	0.0000
0.6	0.216	0.216	0.0000

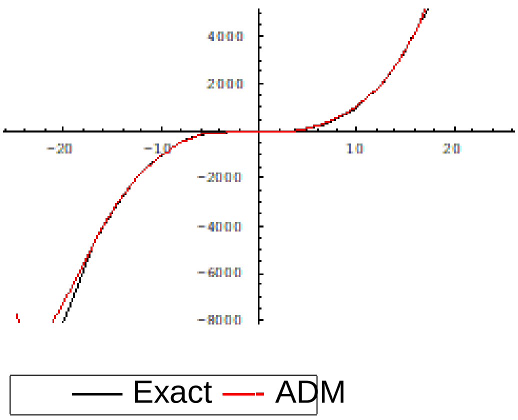

Figure 3: The Approximation solution for ADM and Exact solution

Figure 1. The Approximation for ADM and Exact solution $y = x^3$

Table 1. The difference between the results we have got by (ADM) and exact solution.

Figure 1. We noted that the solutions are approximate, although we have found y_0, y_1, y_2, only.

4. Conclusion

In this study, we gave new differential operators to solve the Van der Pol equation by Adomian Decomposition Method (ADM). (ADM) is powerful and strong to solve the non-linear differential equations, and considered the Van der Pol equation is a model of non-linear differential equations, which we can finding the approximate solutions. We gave us some non-linear examples to explain the Van der Pol equation by (ADM), and we shown that through the tables and graphics convergence the solutions from the exact solution.

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