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Finite Element Analysis of Linear Buckling of Circular Thin Plates for Different Boundary Conditions

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***Abstract*— Linear buckling analysis, which is also called eigenvalue approach, is able to predict the theoretical critical buckling load. This method is used for Finite Element Analysis and also in compliance with theoretical methods in solid mechanics or theory of plates and shells. Material behaviour is completely elastic for this type of analysis and will be processed based on solving eigenvalue problem. This paper presents a finite element model for a circular plate with fixed edge and simply supported edge as two different boundary conditions. The study uses *ABAQUS* (*Student Edition 2019*) software to derive the finite element model of the circular plate. The results obtained through FEM would be compared with an exact solution for both boundary conditions.**

Keywords*— *Critical buckling load; Equilibrium method; Partial differential equation; Finite element method; Exact solution.

1. Introduction

Thin plates of various shapes used in naval and aeronautical structures are often subjected to normal compressive and shearing loads acting in the middle plane of the plate (in-plane loads). Under certain conditions such loads can result in a plate buckling. Buckling or elastic instability of plates is of great practical importance [1-5].

The buckling load depends on the plate thickness: the thinner the plate, the lower is the buckling load. In many cases, a failure of thin plate elements may be attributed to an elastic instability and not to the lack of their strength. Therefore, plate buckling analysis presents an integral part of the general analysis of a structure [1,2,3].

Finite element analysis is a numerical approach for different types of analyses including static and dynamic analyses. Pouladkhan et al. [6] studied a finite element model for a simply supported and simply supported-simply supported-fixed-free rectangular thin plate for buckling analysis using ABAQUS software. Pouladkhan et al. [7] presented a finite element model for a simply supported rectangular thin plate for vibration analysis using ABAQUS software. A finite element model for a sandwich plate for deflection and stress analysis using ABAQUS software were investigated by Pouladkhan et al. [8].

In this study, we consider a systematic but simplified analysis of plate buckling and obtain some useful relations between the critical loads and plate parameters for the corresponding exact solutions. The exact solutions will be compared with finite element analysis for different boundary conditions.

2. General postulations of the theory of stability of plates

This section contains some fundamentals of classical stability analysis of thin elastic plates. It should be noted that the stability analysis of plates is qualitatively similar to the Euler stability analysis of columns [2].

Consider an ideal thin, elastic plate, which is assumed initially to be perfectly flat and subjected to external in-plane compressive and shear loads acting strictly in the middle of plane of the plate. The resulting deformations of this plate are characterized by the absence of deflections (□≠0, □≠0, □□□ □=0) and, consequently, of the bending and twisting moments, as well as the transverse shear forces. Such a plane stress condition of the plate is referred to as an initial or flat configuration of equilibrium, assuming the equilibrium conditions between applied external loads and the corresponding in-plane stress resultants [6].

Depending mainly on values of the applied in-plane loads, an initial, flat configuration of plate equilibrium may be stable or unstable. The initial configuration of elastic equilibrium is *stable*, if when the plate is displaced form this equilibrium state by an infinitesimal disturbance, say a small lateral force, the deflected plate will tend to come back to its initial, flat configuration when the disturbance is removed. The initial configuration of equilibrium is said to be *unstable*, if when the plate is displaced from this equilibrium position by a small lateral load, it will tend to displace still further when the load is removed. If the plate remains at the displaced position even after the small lateral load is removed, it is said to be in *neutral equilibrium*. Thus, the plate in neutral equilibrium is neither stable nor unstable [6].

The goal of the buckling analysis of plates is to determine the critical buckling loads and the corresponding buckled configuration of equilibrium. We consider below the linear buckling analysis of plates based on the following assumptions:

(a) Prior to loading, a plate is ideally flat and the entire applied external loads act strictly in the middle plane of the plate.
(b) States of stress is described by equations of the linear plane elasticity. Any changes in the plate dimensions are neglected prior to buckling.
(c) All the loads applied to the plate are dead loads; that is, they are not changed either in magnitude or in direction when the plate deforms.
(d) The plate bending is described by Kirchhoff 's plate bending theory.

The linear buckling analysis of plates based on these assumptions makes it possible to determine accurately the critical loads, which are of practical importance in the stability analysis of thin plates. However, this analysis gives no way of describing the behavior of plates after buckling, which is also of considerable interest. The post-buckling analysis of plates is usually difficult because it is basically a nonlinear problem.

Classical buckling problems of plates can be formulated using (1) the equilibrium method, (2) the energy method, and (3) the dynamic method. In this study, we use the equilibrium method [2,3,4].

3. The equilibrium method

Consider an initial state of equilibrium of a plate subjected to the external edge loads acting in the middle plane of the plate. Let the corresponding in-plane stress resultants in this initial state be N_x, N_y, and N_{xy}. They may be found from the solution of the plane stress problem for the given plate geometry and in-plane external loading. For the plate, the in-plane external edge loads that result in an elastic instability as in the case of a beam column, are independent of the lateral loads. Therefore, the governing differential equation of the linear buckling analysis of plates can be presented as follow:

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y} + N_y\frac{\partial^2 w}{\partial y^2}\right) \quad (1)$$

Where N_x, N_y, and N_{xy} are the internal forces acting in the middle surface of the plate due to the applied in-plane loading. This equation is a homogeneous, partial differential equation. The mathematical problem is to solve this equation with appropriate homogeneous boundary conditions. In general, such a problem has only a trivial solution corresponding to the initial, flat configuration of equilibrium (i.e. w=0). However, the coefficients of the governing equation depend on the magnitudes of the stress resultants, which are in turn, connected with the applied in-plane external forces, and we can find values of these loads for which a nontrivial solution is possible. The smallest value of these loads will correspond to a critical load [3,4].

4. Buckling of circular plates

Circular plates in some measuring instruments are used as sensitive elements reacting to a change in the lateral pressure. In some cases – in temperature changes, in the process of their assembly – these elements are subjected to the action of radial compressive forces from a supporting structure. As a result, buckling of the circular plates can take place [9,10].

Let us consider a circular solid plate subjected to uniformly distributed in-plane compressive radial forces q_r, as shown in Fig. 1. We confine our buckling analysis to considering only axisymmetric configurations of equilibrium for the plate [11,12]. We can use the polar coordinates r and φ to transfer the governing differential equation of plate buckling (Eq. (1)), derived for a rectangular plate, to a circular plate [13].

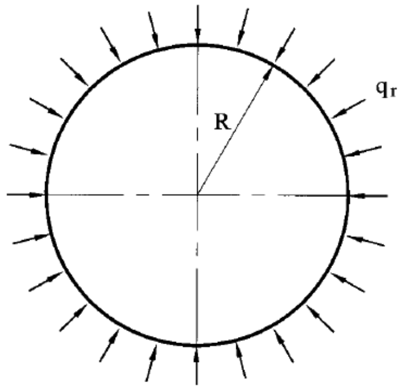


Fig. 1. Circular solid plate subjected to uniformly distributed in-plane compressive radial forces [1].

For the particular case of axisymmetric loading and equilibrium configurations, we have

$$N_x = N_y = N_r = -q_r \quad , \qquad N_{xy} = 0 \quad (2)$$

Denoting

$$\mu^2 = \frac{q_r}{D} \quad (3)$$

and using the relations between the polar and Cartesian coordinates, we obtain the following differential equation of the axisymmetrically loaded circular plate subjected to in-plane compressive forces q_r:

$$\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}+\mu^2\left[\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right]=0 \quad (4)$$

Let us introduce the following new variable:

$$\rho=\mu r, \quad (5)$$

which represents a dimensionless polar radius. Using the new variable ρ, we can rewrite Eq. (4), as follows:

$$\frac{d^4w}{d\rho^4}+\frac{2}{\rho}\frac{d^3w}{d\rho^3}+\left(1-\frac{1}{\rho^2}\right)\frac{d^2w}{d\rho^2}+\frac{1}{\rho}\left(1+\frac{1}{\rho^2}\right)\frac{dw}{d\rho}=0 \quad (6)$$

This is a fourth-order linear, homogeneous differential equation. The general solution of this equation is given by [2] as

$$w(\rho)=C_1+C_2\ln\rho+C_3J_0(\rho)+C_4Y_0(\rho), \quad (7)$$

where $J_0(\rho)$ and $Y_0(\rho)$ are the Bessel functions of the first and second kind of zero orders, respectively. They are tabulated in Ref. [14]. In Eq. (7), C_i $(i=1,\cdots,4)$ are constants of integration. Since $w(\rho)$ must be finite for all values of ρ, including $\rho=0$, then the two terms $\ln\rho$ and $Y_0(\rho)$, having singularities at $\rho=0$, must be dropped for the solid plate because they approach an infinity when $\rho\to\infty$. Thus, for the solid circular plate, Eq. (7) must be taken in the form

$$w(\rho)=C_1+C_3J_0(\rho) \quad (8)$$

Now, we want to determine the critical values of the radial compressive forces, q_r, applied to the middle plane of solid circular plates for two types of boundary supports.

4.1. Circular plate with fixed edge

Let the radius of the plate be *R*. We denote the corresponding value of μR by β, i.e., $\beta=\mu R$. The boundary conditions are

$$w(\beta)=0|\rho=\beta,\quad \vartheta(\beta)=0|\rho=\beta, \quad (a)$$

where the slope of the plate midsurface, $\vartheta(\rho)$, is given by

$$\vartheta(\rho)=\mu\frac{dw}{d\rho}=\mu C_3\frac{d}{d\rho}J_0(\rho) \quad (b)$$

From the Bessel function theory [14], that it follows

$$J_1(\rho)=-\frac{d}{d\rho}J_0(\rho) \quad (c)$$

Thus, we can write the following representations for the slope

$$\vartheta(\rho)=-\mu C_3J_1(\rho), \quad (9)$$

and

$$\vartheta(\beta)=-\mu C_3J_1(\beta) \quad \text{on the boundary}, \quad (10)$$

where $J_1(\,)$ is the Bessel function of the first kind of the first order. Substituting the expressions (9) and (10) into the boundary conditions (a) yields the following system of two linear homogeneous equations:

$$C_1+C_3J_0(\beta)=0,$$
$$-\mu C_3J_1(\beta)=0$$

For a nontrivial solution of these equations:

$$J_1(\beta)=0$$

From the tables of roots of the Bessel functions [14] it follows that the smallest root of the function $J_1(\beta)$ is $\beta_{min}=3.8317$. Noting that $\beta^2=(\mu R)^2=q_r/D\,R^2$, we obtain the critical value of the compressive forces as

$$q_{r,cr}=(3.8317)^2\frac{D}{R^2}=14.68\frac{D}{R^2} \quad (11)$$

For a steel plate with the following geometric and mechanical parameters:

$E=200\ \ Gpa\,;\vartheta=0.3\,;h=1\ mm\,;R=0.1m$

$D=\frac{Eh^3}{12(1-\vartheta^2)}\qquad D=18.315\ \ N.m^2$

$q_{r,cr}=14.68\frac{D}{R^2}=14.68\frac{18.315}{0.1^2}=26886.42\ \ N/m$

4.2. Circular plate with simply supported edge

The boundary conditions for this type of support are

$$w(\beta)=0|_{\rho=\beta}\,,M_r(\beta)=0|_{\rho=\beta} \quad (d)$$

The radial bending moment, M_r, for an axisymmetrically loaded circular plate is given by Eq. (12).

$$M_r=-D\left(\frac{d^2w}{dr^2}+\frac{\vartheta}{r}\frac{dw}{dr}\right) \quad (12)$$

When passing from variable r to the variable ρ, the expression for M_r becomes

$$M_r=-D\mu^2\left(\frac{d^2w}{d\rho^2}+\frac{\vartheta}{\rho}\frac{dw}{d\rho}\right) \quad (13)$$

Using Eq. (8) for the deflections and Eq. (13) for the radial bending moments, we can represent the second boundary condition (d) in the form

$$-D\mu^2\left[\frac{d^2}{d\rho^2}J_0(\rho)+\frac{\vartheta}{\rho}\frac{d}{d\rho}J_0(\rho)\right]=0|_{\rho=\beta} \quad (14)$$

Using the relationships between the Bessel functions of the first kind [14], we have

$$\frac{d^2}{d\rho^2}J_0(\rho)=-J_0(\rho)+\frac{1}{\rho}J_1(\rho) \quad (e)$$

Substituting the expression for the deflections (8) into the boundary conditions (d) and taking into account Eqs. (14) and (e), we arrive at the following system of linear homogeneous equations:

$$C_1 + C_3 J_0(\beta) = 0$$
$$-D\mu^2 C_3[\beta J_0(\beta) - (1 - \vartheta)J_1(\beta)] = 0 \quad \text{(f)}$$

A nontrivial solution of this system of equations leads to the following:

$$\beta J_0(\beta) - (1 - \vartheta)J_1(\beta) = 0 \quad \text{(g)}$$

Letting $\vartheta = 0.3$ and using the tables of the Bessel function [14], we can determine the smallest nonzero root of Eq. (g). We have

$\beta_{min} = 2.0485$,

and the critical value of an intensity of the radial compressive forces is

$$q_{r,cr} = 4.196\frac{D}{R^2} \quad (15)$$

For a steel plate with the following geometric and mechanical parameters:

$E = 200\ Gpa\,; \vartheta = 0.3\,; h = 1\ mm\,; R = 0.1m$

$$D = \frac{Eh^3}{12(1 - \vartheta^2)} \qquad D = 18.315\ N.m^2$$

$$q_{r,cr} = 4.196\frac{D}{R^2} = 4.196\frac{18.315}{0.1^2} = 7684.97\ N/m$$

Comparing the values of the critical compressive forces for the clamped and simply supported circular solid plates, we can conclude that the replacement of the supported edges with clamped ones increases the critical force by a factor of 3.5.

5. The Finite Element Method (FEM)

The finite element method (FEM) is based on the concept that one can replace any continuum by an assemblage of simply shaped elements with well-defined force displacement and material relationships. While one may not be able to derive a closed-form solution for the continuum, one can derive an approximate solution for the element assemblage that replaced it.

According to the FEM, a plate is discretized into a finite number of elements (usually, triangular or rectangular in shape), called *finite elements* and connected at their nodes and along interelement boundaries. Unknown functions (deflections, slopes, internal forces, and moments) are assigned in the form of undetermined parameters at those nodes. The equilibrium and compatibility conditions must be satisfied at each node and along the boundaries between finite elements [15].

In this study a comparison between the FEM approach with exact solution based on the equilibrium method has been investigated and Mesh Convergence Curve criterion is considered to optimize the FEM results. For this investigation, ABAQUS software has been employed to derive the finite element model of the circular plate. The Eigenvalue Method is used for finite element analysis of linear buckling of the circular plate.

6. Geometry and problem description

6.1. Circular plate with fixed edge

The model used for this study is a circular plate with fixed edge, which has been discretized by S8R elements, S8R: An 8-node doubly curved thick shell, reduced integration [16]. Boundary configuration and typical finite element model of the circular plate with fixed edge are shown in Figs. 2 and 3 respectively. Table 1 shows number of elements used to achieve optimum mesh for the circular plate with fixed edge.

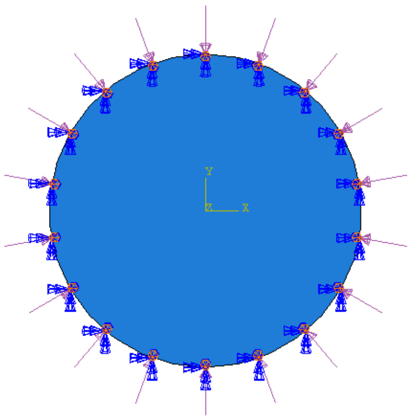


Fig. 2. Boundary configuration of the circular plate with fixed edge.

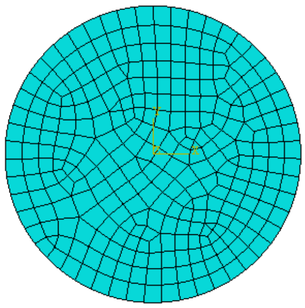


Fig. 3. Typical finite element model of the circular plate with fixed edge.

Table 1. Number of elements used to achieve optimum mesh of the circular plate with fixed edge.

A.G.S	Number of Mesh	Critical Load ($q_{r,cr}\ ;\ N/m$)
0.02	110	27752
0.0125	291	26975
0.0120	309	26982
0.0150	199	27097
0.0140	216	27033
0.0130	259	27014
0.0121	309	26982
0.0124	298	26968
0.0123	298	26968
0.0122	309	26982

Where A.G.S is Approximate Global Size. Based on the table, the buckling critical load for the circular plate from Finite Element analysis compared to the exact solution can be obtained when A.G.S is 0.0123 and Number of Elements equals to 298. In this case, the Critical Load $q_{r,cr} = 26968\ N/m$. According to the *Mesh Convergence* criterion, the above mesh (298 elements) is the optimum one, since the error is minimum and we have:

$$\frac{26982 - 26968}{26968} \times 100 = 0.0519\% < 5\%$$

Fig. 4 illustrates the *Mesh Convergence Curve* from finite element analysis of the circular plate with fixed edge.

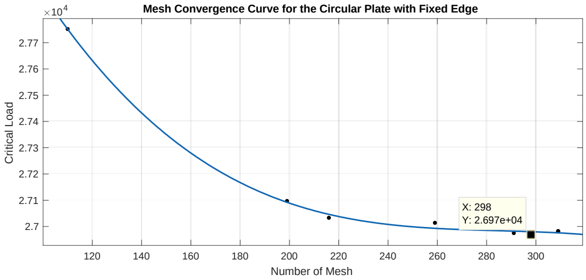
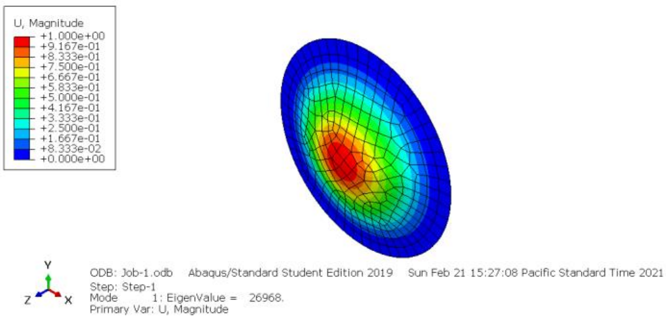


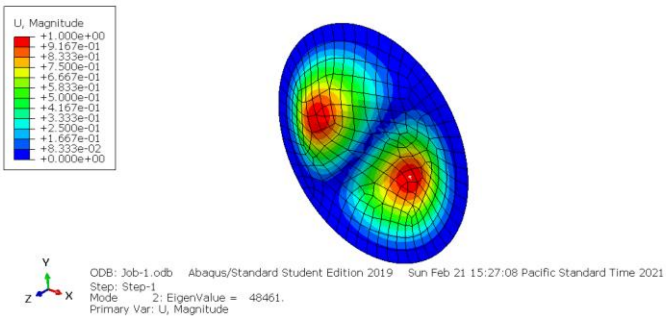
Fig. 4. Mesh convergence curve for the finite element model of the circular plate with fixed edge.

The first 5 mode shapes for the buckled circular plate are shown in the following figures, Fig. 5. It is clear that by increasing mode number, the Eigenvalue (Critical Load) is increased.

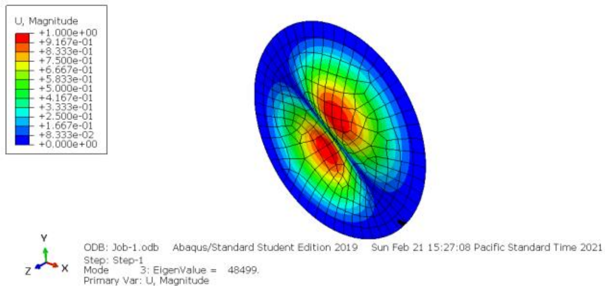
Mode Shape 1:



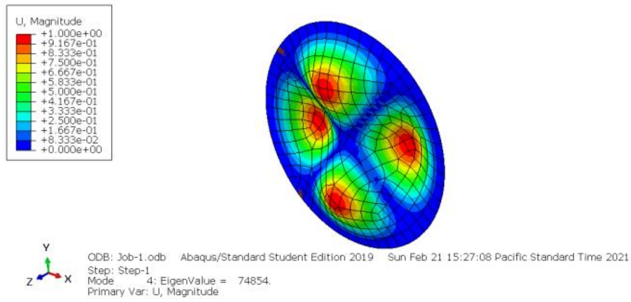
Mode Shape 2:



Mode Shape 3:



Mode Shape 4:



Mode Shape 5:

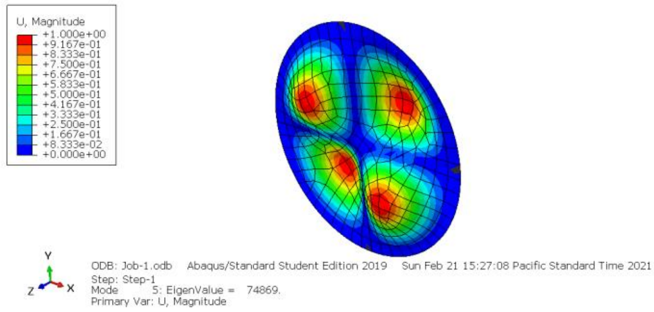


Fig. 5. Mode shapes and critical loads of the circular plate with fixed edge.

6.2. Circular plate with simply supported edge

The model used for this study is a circular plate with simply supported edge, which has been discretized by S8R elements, S8R: An 8-node doubly curved thick shell, reduced integration [16]. Boundary configuration and typical finite element model of the circular plate with simply supported edge are shown in Figs. 6 and 7 respectively. Table 2 shows number of elements used to achieve optimum mesh for the circular plate with simply supported edge.

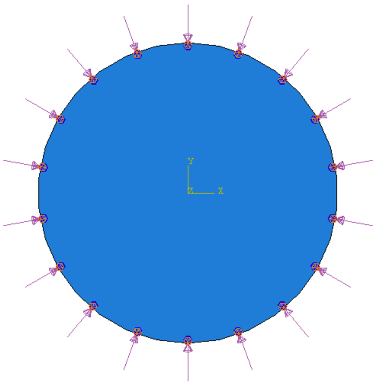


Fig. 6. Boundary configuration of the circular plate with simply supported edge.

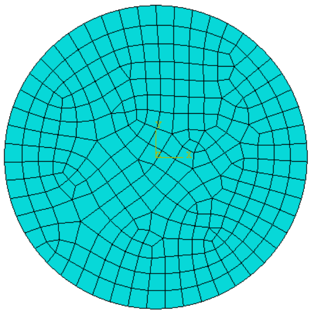


Fig. 7. Typical finite element model of the circular plate with simply supported edge.

Table 2. Number of elements used to achieve optimum mesh of the circular plate with simply supported edge.

A.G.S	Number of Mesh	Critical Load ($q_{r,cr}$; N/m)
0.0125	291	7692.9
0.0124	298	7692.4
0.0123	298	7692.4
0.0122	309	7698.6
0.0121	309	7698.6
0.0120	309	7698.6
0.0130	259	7697.2
0.0140	216	7693.1
0.0150	199	7697.3

Where A.G.S is Approximate Global Size. Based on the table, the buckling critical load for the circular plate from Finite Element analysis compared to the exact solution can be obtained when A.G.S is 0.0123 and Number of Elements equals to 298. In this case, the Critical Load $q_{r,cr} = 7692.4\ N/m$. According to the *Mesh Convergence* criterion, the above mesh (298 elements) is the optimum one, since the error is minimum and we have:

$$\frac{7698.6 - 7692.4}{7692.4} \times 100 = 0.0806\% < 5\%$$

Fig. 8 illustrates the *Mesh Convergence Curve* from finite element analysis of the circular plate with simply supported edge.

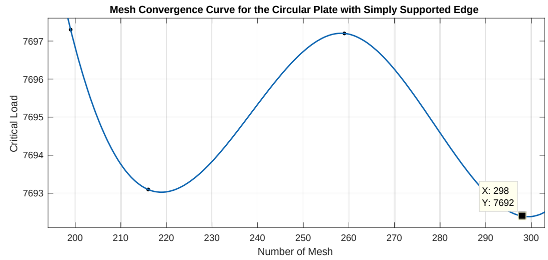
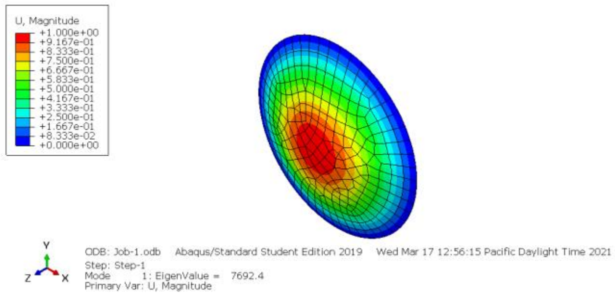


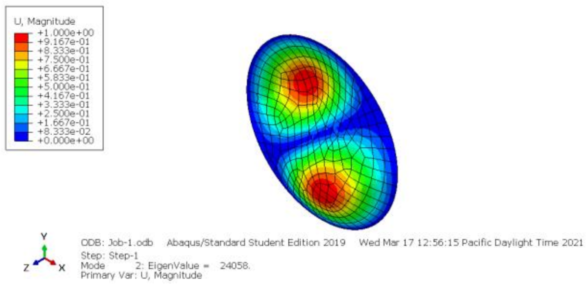
Fig. 8. Mesh convergence curve for the finite element model of the circular plate with simply supported edge.

The first 5 mode shapes for the buckled circular plate are shown in the following figures, Fig. 9. It is clear that by increasing mode number, the Eigenvalue (Critical Load) is increased.

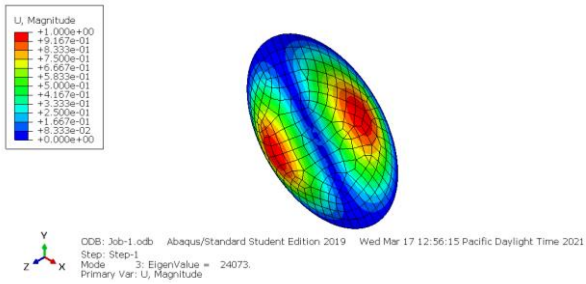
Mode Shape 1:



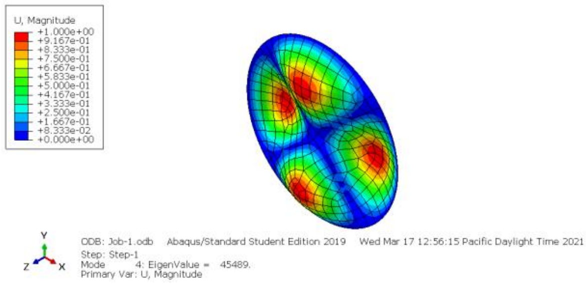
Mode Shape 2:



Mode Shape 3:



Mode Shape 4:



Mode Shape 5:

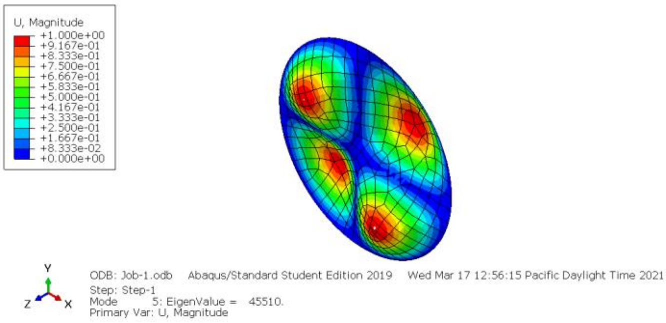


Fig. 9. Mode shapes and critical loads of the circular plate with simply supported edge.

7. Conclusion

A finite element model was presented for this study. This paper reviewed the capability of the shell element (S8R) provided by commercialized FEA codes, and discussed a simple case of static finite element analysis. Based on the finite element modeling technique, the study showed admissible results in comparison with exact solutions for a circular plate with fixed edge and simply supported edge. For both boundary conditions, the *Critical Buckling Load* was investigated and obtained an acceptable result from finite element analysis. Comparing the values of the critical compressive forces for the clamped and simply supported circular solid plates, we can conclude that the replacement of the supported edges with clamped ones increases the critical force by a factor of 3.5. Based on the finite element modeling, we are able to simulate more complicated models in *ABAQUS*, which is difficult to present exact solutions for them to predict the critical buckling load.

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