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# Fundamental Natural Frequency Analysis of Axisymmetric Vibrations of a Solid Circular Plate by using Ritz and FEM Methods

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***Abstract*— This paper presents an approximate solution and a finite element model for free flexural vibrations of a circular plate with fixed edge. The natural frequencies and mode shapes of the circular plate can be determined by Modal analysis. The Ritz method has been employed as a direct numerical method of approximating eigenvalue. The study uses *ABAQUS* (*Student Edition 2019*) software to derive the finite element model of the circular plate. The results obtained through FEM would be compared with the Ritz method for the clamped circular plate.**

***Keywords— Ritz method; Finite element method; Modal analysis; Natural frequencies; Mode shapes.***

## 1. Introduction

In engineering practice, however, many components of machines and structures are subjected to dynamic effects, produced by time-dependent external forces or displacements [1].

Moving vehicles, wind gusts, seismic disturbances, unbalanced machine vibrations, flight loads, sound, etc., may create dynamic loads. Dynamic effects of time-dependent loads on structures are studied in structural dynamics. Structural dynamics deals with time-dependent motions of structures, primarily, with vibration of structures, and analyses of the internal forces associated with them. Thus, its objective is to determine the effect of vibrations on the performance of the structure or machine [2].

The dynamics of plates, which are continuous elastic systems, can be modeled mathematically by partial differential equations based on Newton's laws or by integral equations based on the considerations of virtual work. In practical applications only the lateral vibration is of interest, and the effects of extensional vibrations in the middle plane may be neglected. Therefore, the inertia forces, associated with the lateral translation of the plate, are considered. In this paper, only the simplified theory of plate vibrations is introduced; some physical phenomena, associated with, for instance, damping effects, are not considered [2,3].

Damping effects are caused either by internal friction or by the surrounding media. Although structural damping is theoretically present in all plate vibrations, it has usually little or no effect on (a) the natural frequencies and (b) the steady-state amplitudes; consequently, it can be safely ignored in the initial treatment of the problem [4].

The derivation of the governing differential equation of motion is, in most cases, a simple extension of the static case by adding effective forces to the plate that result from accelerations of the mass of the plate. These are the inertia forces.

We consider various kinds of motion of plates. There is a *free vibration*, which occurs in the absence of applied loads but may be initiated by applying initial conditions to the plate. The free vibration deals with natural characteristics of the plates, and these natural vibrations occur at discrete frequencies, depending only on the geometry and material of the plates. Then, there is a *forced vibration*, which results from an application of time-dependent loads. Forced vibrations come in two kinds: a harmonic response, when a periodic force is applied to the plate; and a transient response, when the applied force is not a periodic force [5].

The fundamental natural frequency for vibration analysis of many structural systems, including plates and shells can be implemented by FEM approach. Finite element analysis is a numerical method for different types of analyses, including static and dynamic analyses. The commercial FEA packages such as ABAQUS gives convenient access to perform both static and dynamic analyses for structural elements.

Pouladkhan et al. [6] studied a finite element model for a simply supported and simply supported-simply supported-fixed-free rectangular thin plate for buckling analysis using ABAQUS software. Pouladkhan et al. [7] presented a finite element model for a simply supported rectangular thin plate for vibration analysis using ABAQUS software. A finite element model for a sandwich plate for deflection and stress analysis using ABAQUS software were investigated by Pouladkhan et al. [8]. Pouladkhan et al. [9] presented an exact solution and a finite element method (using ABAQUS) for a smart piezoelectric ceramic rod under static load. Pouladkhan [10] presented a finite element model using ABAQUS for buckling analysis of a circular plate with fixed edge and simply supported edge. A finite element model for electrostatic analysis of a smart piezoceramic plate by using ABAQUS was investigated by Pouladkhan [11].

In this study, we will consider a systematic but simplified analysis of flexural vibrations of a circular plate and obtain some useful relations between the fundamental natural frequency of axisymmetric vibrations and plate parameters for the Ritz method. The results obtained through FEM will be compared with the Ritz method for the clamped circular plate.

## 2. Free flexural vibrations of circular plates

Let us consider a freely vibrating, solid, circular plate of radius *a*, having a constant thickness *h*. Using the polar coordinates, *r* and $\varphi$, with the origin at the center of the plate, we can rewrite the governing differential equation of the free vibration of plates, Eq. (1), as follows [1]:

$$D\nabla^2\nabla^2 w(x,y,t) + \rho h \frac{\partial^2 w}{\partial t^2}(x,y,t) = 0 \tag{1}$$

$$D\nabla_r^2\nabla_r^2 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{2}$$

Where $\nabla_r^2$ is the Laplace operator. Assume that the deflection of the middle surface of the plate can be approximated as

$$w(r,\varphi,t) = W(r,\varphi)F(t) \tag{3}$$

Introducing the above into Eq. (2), yields

$$DF(t)\nabla_r^2\nabla_r^2 W + \rho h W \frac{d^2F}{dt^2} = 0 \;\; or \;\; \frac{D\nabla_r^2\nabla_r^2 W}{\rho h W} = -\frac{\frac{d^2F}{dt^2}}{F} \tag{4}$$

Since the left-hand side of this equation is a function of variables r and $\varphi$ whereas the right-hand side depends only on time variable t, we can conclude that the ratios in the left- and right-hand sides of Eq. (4) must be constant. Denote the aforementioned constant ratio on the right-hand side of Eq. (4) by $\omega^2$, i.e.,

$$\frac{d^2F}{dt^2} = -\omega^2 F \tag{5}$$

Where $\omega$ is the *natural frequency of vibrations*. Solving this for *F*, yields

$$F = A\sin(\omega t + \varphi_0) \tag{6}$$

Where $\varphi_0$ is an arbitrary constant. The shape function $W(r,\varphi)$ satisfies the differential equation

$$\frac{D\nabla_r^2\nabla_r^2 W}{\rho h W} = \omega^2 \;\; or \;\; \nabla_r^2\nabla_r^2 W - \lambda^4 W = 0 \tag{7}$$

Where

$$\lambda^4 = \frac{\omega^2 \rho h}{D} \tag{8}$$

Let us go from the variable *r* to the dimensionless variable $\zeta = \lambda r$. Then, Eq. (7) becomes

$$\left(\frac{\partial^2}{\partial\zeta^2} + \frac{1}{\zeta}\frac{\partial}{\partial\zeta} + \frac{1}{\zeta^2}\frac{\partial^2}{\partial\varphi^2}\right)^2 W - W = 0 \tag{9}$$

Its solution is of the following form [12,13]:

$$W(r,\varphi) = [C_1 J_n(\zeta) + C_2 I_n(\zeta) + C_3 Y_n(\zeta) + C_4 K_n(\zeta)]\sin(n\varphi + \alpha) \tag{10}$$

Where $n = 0, 1, \cdots, \infty$; $C_1, \cdots, C_4$ are constants of integration; and $J_n()$, $I_n()$, $Y_n()$, and $K_n()$ are Bessel functions of the first and second kind of the real and imaginary arguments, respectively [12,13], and $\alpha$ is a constant. Since the origin of the polar coordinate system is taken to coincide with the center of the circular plate having no internal holes or supports at the center, the terms $Y_n(\zeta)$ and $K_n(\zeta)$ must be discarded to avoid infinite deflections and stresses at $r = 0$. When these simplifications are employed, Eq. (10) becomes, for a typical mode,

$$W = [C_1 J_n(\zeta) + C_2 I_n(\zeta)]\sin(n\varphi + \alpha) \tag{11}$$

Assume that the plate is clamped along its contour. The boundary conditions are

$$W = \frac{\partial W}{\partial r} = 0|_{r=a} \tag{12}$$

When Eq. (11) is substituted into the above boundary conditions, the existence of a nontrivial solution yields the following characteristic determinant:

$$\begin{vmatrix} J_n(\zeta) & I_n(\zeta) \\ J_n'(\zeta) & I_n'(\zeta) \end{vmatrix} = 0 \tag{13}$$

Where the primes are used to indicate a differentiation with respect to the argument, in this case to $\zeta$. Using the following recursion relationships [12,13]:

$$\begin{aligned} \zeta J_n'(\zeta) &= nJ_n(\zeta) - \zeta J_{n+1}(\zeta) \\ \zeta I_n'(\zeta) &= nI_n(\zeta) + \zeta I_{n+1}(\zeta) \end{aligned} \tag{14}$$

And expanding Eq. (13) gives

$$J_n(\zeta)I_{n+1}(\zeta) + I_n(\zeta)J_{n+1}(\zeta) = 0 \tag{15}$$

The eigenvalues $\zeta$ determining the frequencies $\omega$ are the roots of Eq. (15). The Bessel functions are widely tabulated for small values of *n* [12]. For circular plates simply supported all around ($w = 0$ and $M_r = 0$), the frequency equation is of the form

$$\frac{J_{n+1}(\zeta)}{J_n(\zeta)} + \frac{I_{n+1}(\zeta)}{I_n(\zeta)} = \frac{2\zeta}{1-\vartheta} \tag{16}$$

If a plate edge is completely free ($M_r = 0$ and $V_r = 0$), then the frequency equation can be represented as follows (for $\zeta \gg n$) [14]:

$$\frac{J_n(\zeta)}{J_n'(\zeta)} \cong \frac{[\zeta^2 + 2(1-\vartheta)n^2]\left[\frac{I_n(\zeta)}{I_n'(\zeta)}\right] - 2\zeta(1-\zeta)}{\zeta^2 - 2(1-\vartheta)n^2} \tag{17}$$

The natural frequencies and pertinent mode shapes for solid circular plates can be also calculated by using the Ritz or Galerkin methods.

## 3. Fundamental natural frequency of axisymmetric vibrations of a circular plate

Consider a solid circular plate with radius *R*, which is clamped along its boundary. Let us determine the fundamental natural frequency of axisymmetric vibrations of the plate. We can take the shape function in the first approximation as follows:

$$W = C(R^2 - r^2)^2 \tag{18}$$

Evidently, the above expression satisfies the given boundary conditions ($\mathrm{w} = 0$ and $\partial \mathrm{w}/\partial \mathrm{r} = 0$) exactly. We apply the Ritz method. It can be shown that for circular plates with a fastened edge the expression for the potential energy becomes simplified:

$$U_{max} = \iint_A \frac{D}{2}\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \varphi^2}\right)^2 r dr d\varphi \tag{19}$$

The corresponding expression for the kinetic energy in polar coordinates has the form

$$K_{max} = \frac{\omega^2}{2}\iint_A \rho h W^2(r,\varphi) r dr d\varphi \tag{20}$$

Substituting for *W* from Eq. (18) into Eqs. (19) and (20), and evaluating the corresponding integrals, one obtains

$$U_{max} = C^2 D \frac{\pi}{2} R^6 \frac{32}{3}$$
$$K_{max} = \frac{\omega^2}{2} C^2 \pi R^{10} \frac{1}{10}\rho h \tag{21}$$

Then, *the fundamental natural frequency* (the first approximation) may be found from *Rayleigh's principle* for finding the lowest natural frequency of a vibrating plate, which is of great interest in applied vibration analysis. This principle is based on the following statement: if the vibrating system is conservative (no energy is added or lost), then the maximum kinetic energy, $K_{max}$, must be equal to the maximum potential (strain) energy, $U_{max.}$ Applying this principle, we consider an elastic plate undergoing free vibrations with the fundamental mode as a system with one degree of freedom. Taking into account that only free flexural vibrations are of interest, we can present the above principle as follows:

$$U_{max} = K_{max} \tag{22}$$

Therefore, *the fundamental natural frequency* can be found as follows:

$$\omega_{11} = \frac{10.33}{R^2}\sqrt{\frac{D}{\rho h}} \tag{23}$$

Notice that the second approximation of this frequency differs from the above value by 1.175% only.

For a steel plate with the following geometric and mechanical parameters:

$$\rho = 7800\ kg/m^3 ; E = 200\ GPa ; \vartheta = 0.3 ; h = 1\ mm ; R = 0.1\ m$$

$$D = \frac{Eh^3}{12(1-\vartheta^2)} \qquad D = 18.315\ N.m^2$$

$$\omega_{11} = \frac{10.33}{0.1^2}\sqrt{\frac{18.315}{7800 \times 0.001}} \quad ; \quad \omega_{11} = 1582.911\ rad/sec\ \ (f = 251.928\ Hz)$$

## 4. The Finite Element Method (FEM)

The finite element method (FEM) is based on the concept that one can replace any continuum by an assemblage of simply shaped elements with well-defined force displacement and material relationships. While one may not be able to derive a closed-form solution for the continuum, one can derive an approximate solution for the element assemblage that replaced it.

According to the FEM, a plate is discretized into a finite number of elements (usually, triangular or rectangular in shape), called *finite elements* and connected at their nodes and along interelement boundaries. Unknown functions (deflections, slopes, internal forces, and moments) are assigned in the form of undetermined parameters at those nodes. The equilibrium and compatibility conditions must be satisfied at each node and along the boundaries between finite elements [15].

In this study a comparison between the FEM approach with the Ritz method for the clamped circular plate has been investigated and Mesh Convergence Curve criterion is considered to optimize the FEM results. For this investigation, *ABAQUS* software has been employed to derive the finite element model of the circular plate. The natural frequencies and mode shapes of the circular plate can be determined by Modal analysis for the finite element method.

## 5. Modal analysis

The frequency extraction procedure for Modal analysis can be described as follows:

- Performs eigenvalue extraction to calculate the natural frequencies and the corresponding mode shapes of a system.

- Will include initial stress and load stiffness effects due to preloads and initial conditions if geometric nonlinearity is accounted for in the base state, so that small vibrations of a preloaded structure can be modeled.

- Is a linear perturbation procedure.

## 6. Eigenvalue extraction

The frequency extraction procedure uses eigenvalue techniques to extract the frequencies of the current system. The eigenvalue problem for the natural frequencies of an undamped finite element model is:

$$(-\omega^2 M + K)\phi = 0 \tag{24}$$

Where

M is the mass matrix (which is symmetric and positive definite).
K is the stiffness matrix (which includes initial stiffness effects if the base state included the effects of nonlinear geometry).
ϕ is the eigenvector (the mode of vibration).

If initial stress effects are not included and there are no rigid body modes, *K* is positive definite; otherwise, it may not be. Negative eigenvalues normally indicate instability.

## 7. Geometry and problem description

The model used for this study is a circular plate with fixed edge, which has been discretized by S4R elements, S4R: A 4-node doubly curved thin or thick shell, reduced integration, hourglass control, finite membrane strains [16]. Boundary configuration and typical finite element model of the circular plate with fixed edge are shown in Figs. 1 and 2 respectively. Table 1 shows number of elements used to achieve optimum mesh for the circular plate with fixed edge.

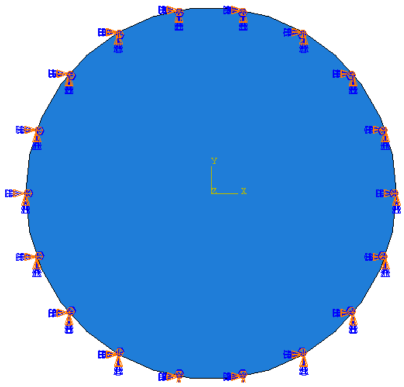


**Fig. 1.** Boundary configuration of the circular plate with fixed edge.

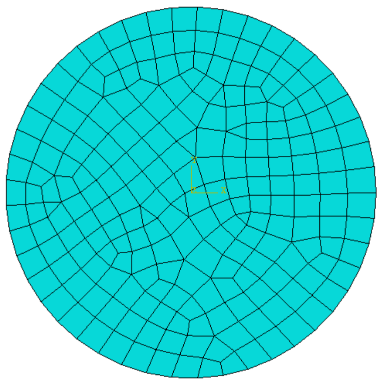


**Fig. 2.** Typical finite element model of the circular plate with fixed edge.

**Table 1.** Number of elements used to achieve optimum mesh of the circular plate with fixed edge.

| A.G.S | Number of Elements | Natural Frequency ($\omega_{11}$ ; $rad/sec$) | Natural Frequency ($f$ ; $Hz$) |
|---|---|---|---|
| 0.028 | 57 | 1645.126 | 261.83 |
| 0.02 | 111 | 1601.898 | 254.95 |
| 0.01 | 429 | 1574.126 | 250.53 |
| 0.015 | 203 | 1587.572 | 252.67 |
| 0.014 | 221 | 1581.038 | 251.63 |
| 0.013 | 256 | 1579.907 | 251.45 |
| 0.012 | 291 | 1578.148 | 251.17 |

Where A.G.S is Approximate Global Size. Based on the table, the fundamental natural frequency for the circular plate from Finite Element analysis compared to the Ritz method can be obtained when A.G.S is 0.014 and Number of Elements equals to 221. In this case, the fundamental natural frequency equals to $\omega_{11} = 1581.038\ \mathrm{rad/sec}$; $f = 251.63\ \mathrm{Hz}$. According to the *Mesh Convergence* criterion, the above mesh (221 elements) is the optimum one, since the error is minimum and we have:

$$\frac{1581.038 - 1579.907}{1579.907} \times 100 = 0.0716\% < 5\%$$

Fig. 3 illustrates the *Mesh Convergence Curve* from finite element analysis of the circular plate with fixed edge.

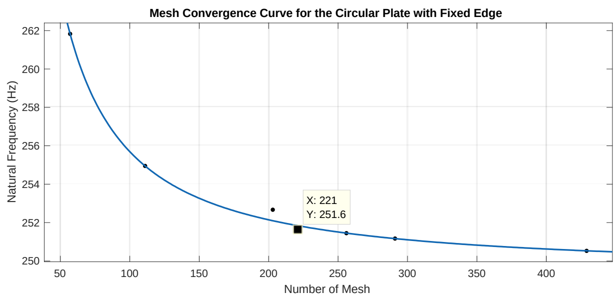
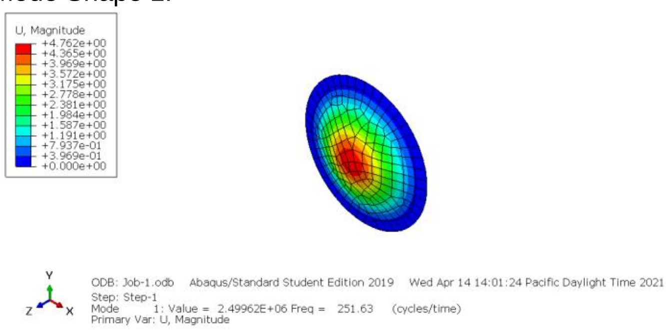


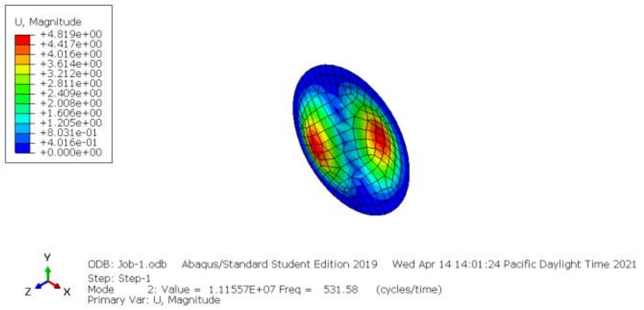

**Fig. 3.** Mesh convergence curve for the finite element model of the circular plate with fixed edge.

The first 10 mode shapes for the vibrated circular plate are shown in the following figures, Fig. 4. It is clear that by increasing mode number, the natural frequency is increased.

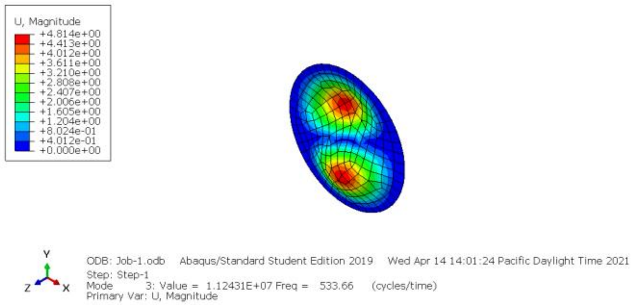
Mode Shape 1:



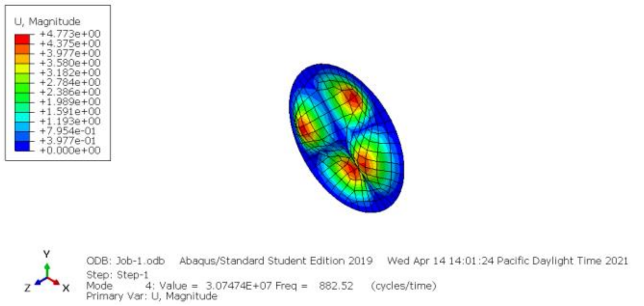
Mode Shape 2:



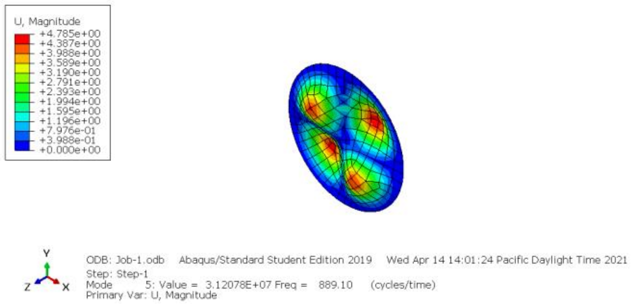

Mode Shape 3:



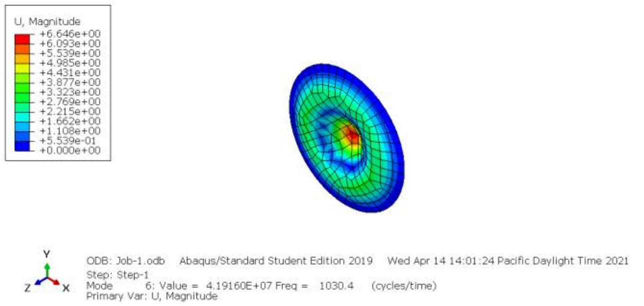

Mode Shape 4:



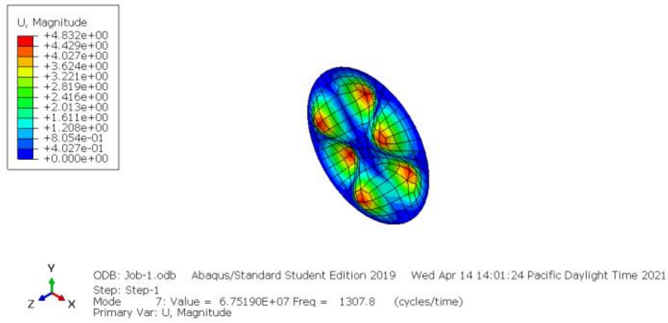

Mode Shape 5:



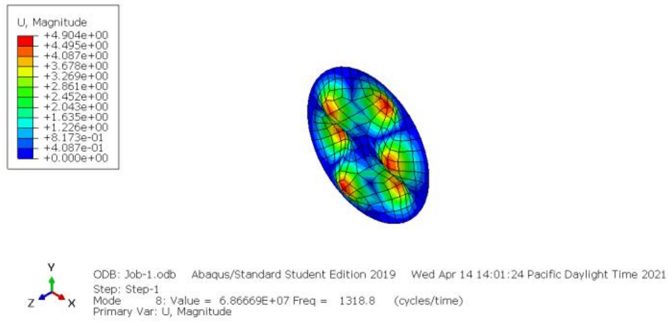

Mode Shape 6:



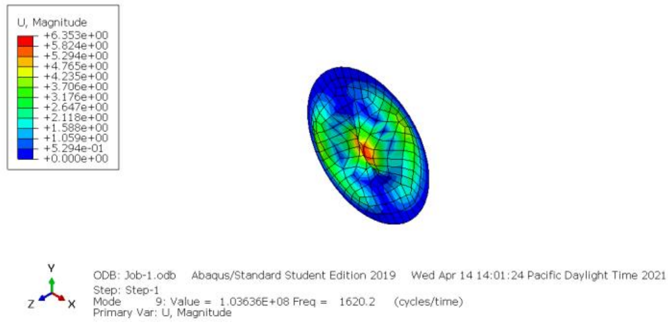

Mode Shape 7:




Mode Shape 8:




Mode Shape 9:




Mode Shape 10:

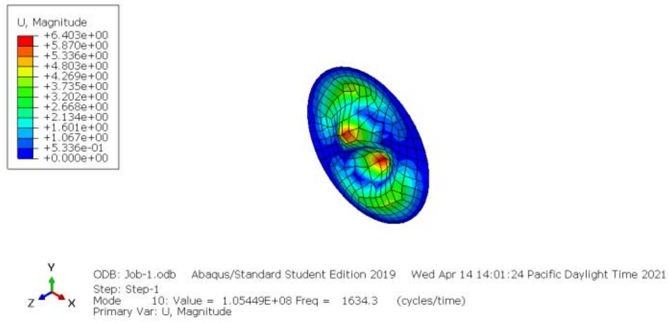



**Fig. 4.** Mode shapes and natural frequencies of the circular plate with fixed edge.

## 8. Conclusion

A finite element model was presented for this study. This paper reviewed the capability of the shell element (S4R) provided by commercialized FEA codes, and discussed a simple case of dynamic finite element analysis. Based on the finite element modeling technique, the study showed admissible results in comparison with the Ritz method for a circular plate with fixed edge. According to the finite element modeling technique, we are able to simulate more complicated models in *ABAQUS*, which is difficult to present exact or approximate solutions for them to predict the fundamental natural frequency.

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