Applied Regula–Falsi Iterative Approach For The Computation Of Eccentricity Anomaly Of Keplerian Orbits

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Abstract*— In this paper, applied Regula–Falsi iterative approach for the computation of eccentricity anomaly of Keplerian orbits was presented. The flowchart and relevant mathematical expressions for the applied Regula–Falsi approach were presented. The two initial guess values required by Regula–Falsi algorithm were determined using analytical models based on the bounds on the value of eccentricity anomaly for any given values of orbital eccentricity and mean anomaly. Some sample computations of eccentricity anomaly were conducted with different combinations of the values of eccentricity (e) and mean anomaly (M). The convergence cycle of the Regula–Falsi algorithm was noted in each case. In all, different combinations of the values of M and e gave rise to different convergence cycles. According to the results, for eccentricity (e =0.99), mean anomaly (M =1°) and tolerance error , ε set in the order of 10^{-15}, the Regula–Falsi algorithm converged after 28 cycles with eccentricity anomaly (E =0.4240960 radians = 24.30°). Similarly, for eccentricity (e =0.5), mean anomaly (M =42°) and tolerance error, ε set in the order of 10^{-15}, the Regula–Falsi algorithm converged after 2 cycles with eccentricity anomaly (E =1.1988490 radians = 68.68°). Furthermore, for eccentricity (e =0.5), mean anomaly (M =42°) , e = 0.9 and tolerance error , ε set in the order of 10^{-15}, the Regula–Falsi algorithm converged after 4 cycles with eccentricity anomaly (E =1.1988490 radians = 68.68°). In all, Regula–Falsi algorithm can effectively be used to compute the eccentricity anomaly of Keplerian orbits. However, the convergence of the algorithm is dependent on the combination of the values of e and M.

Keywords*— *Regula–Falsi Iterative, Numerical Iteration, Eccentricity, Mean Anomaly, Eccentricity Anomaly, Convergence Performance

1. INTRODUCTION

Over the years, Regula–Falsi iteration (RFI) scheme has been used to solve transcendental equations that have no closed-form solutions [1,2,3,4,5,67,8,9]. Basically, Regula–Falsi algorithm requires two initial guess roots, say XL and XU, for the function such that the actual root, say X, is bracketed by the two guess roots, hence, XL ≤ X ≤ XU [10,11,12,13,14,15]. Finding the two initial guess roots that satisfy the root bracketing requirement is always a challenge to users of the classical RFI scheme. In any case, some functions have known upper and lower bounds on the expected root value. In such case, applying the classical RFI scheme becomes easy.

Consequently, in this paper the RFI scheme is applied in the determination of the eccentricity anomaly (E) of Keplerian orbits based on the knowledge of mean anomaly (M) and the orbital eccentricity (e) [16,17,18,19]. Available study showed that for any given values of M and e, the value of E is such that M ≤ E ≤ M + e [20,21,22,23]. In that such case, M and M +e can serve as affective two initial guess roots for the computation of the eccentricity anomaly of Keplerian orbits.

In any case, the number of iterations required for the convergence of numerical iteration schemes reduces as the initial guess roots are closer to the actual roots. In that case, instead of using M as the lower bound on the initial value of E, some published works have presented some other initial start values that are lower than E but also closer to the value of E than the value of M. In this paper, one of such initial guess roots values are used in RFI scheme to determine the eccentricity anomaly for Keplerian orbits. The study also examined the convergence performance of the RFI algorithm under different combinations of the values of e and M.

2. METHODOLOGY

As study presented in [20] stated that for Keplerian orbit with mean anomaly (M) and eccentricity (e), the maximum possible value of the eccentricity anomaly (E) is given as M +e while the lowest value of E is M. However, in [20] , a more efficient initial lower value of E is defined as given in Eq 2. Hence, the two initial guess values for E are, the initial lower value (denoted as $X_{L(0)}$) and the initial upper value (denoted as $X_{U(0)}$), where;

$$X_{U(0)} = M + e \tag{1}$$

$$X_{L(0)} = M + \frac{e(\sin(M))}{1-\sin(M+e)+\sin(M)} \tag{2}$$

These two initial values are then used in the classical Regula–Falsi iteration algorithm to iteratively determine the actual value of E (denoted as $X_{(k)}$) , where k is the iteration cycle counter with value, k =1,2,3,…. Then, for any given cycle, k, the corresponding actual value of E (denoted as $X_{(k)}$) computed by the Regula–Falsi method is given as;

$$x_{(k)} = \frac{x_{L(k-1)} * f(x_{u(k-1)}) - x_{u(k-1)} * f(x_{L(k-1)})}{f(x_{u(k-1)}) - f(x_{L(k-1)})} \quad (3)$$

After $x_{(k)}$ is determined, the $f(x_{(k)})$ is computed and compared with the specified tolerance error, $\boldsymbol{\varepsilon}$; if $f(x_{(k)}) < \boldsymbol{\varepsilon}$ then $x_{(k)}$ is the actual value of E otherwise $x_{L(k)}$ and $x_{U(k)}$ are determined based on the Regula–Falsi algorithm, k is incremented by 1 and $x_{(k)}$ is recomputed. The iteration continues until a value of $x_{(k)}$ for which $f(x_{(k)}) < \boldsymbol{\varepsilon}$ is obtained. The flowchart of the Regula–Falsi algorithm for computing eccentricity anomaly (E) is given in Figure 1.

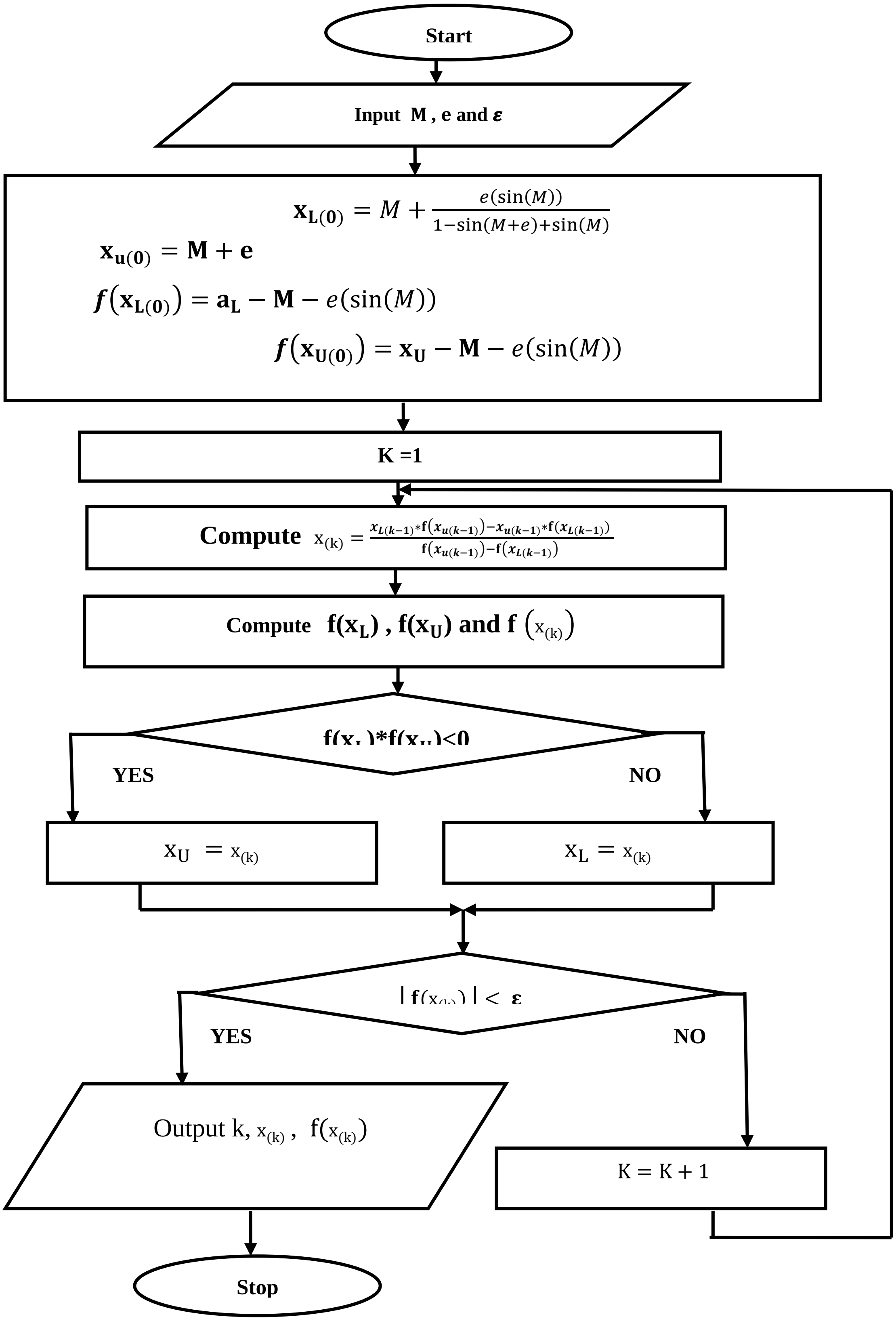

Figure 1 The flowchart for the Regula–Falsi used to determine the eccentricity anomaly (E) of Keplerian orbit

3 RESULTS AND DISCUSSION

The Regula–Falsi flowchart in Figure 1 was implemented in Matlab for different combinations of values of M and e. The results for E for M (°) =1 and e = 0.99 are shown in Table 1 where the computed initial lower and upper values of $x_{L(0)}$ = 0.1179191 radians = 6.76° and $x_{u(0)}$ = 1.0074533 radians = 57.72°.With tolerance error , ε set in the order of 10^{-15}, the results in Table 1 show that it took about 28 Regula Fasi iterations for the actual value of eccentricity (E) =0.4240960 radians = 24.30° to be obtained with error of 8.66E-15 radians.

The results for E for M (°) =42 and e = 0.5 are shown in Table 2 where the computed initial lower and upper values of $\mathbf{x_{L(0)}}$ = 1.1941066radians = 68.41° and $\mathbf{x_{u(0)}}$ = 1.2330383 radians = 70.64°.With tolerance error , ε set in the order of 10^{-15}, the results in Table 2 show that it took about 3 Regula Fasi iterations for the actual value of eccentricity (E) =1.1988490 radians = 68.68° to be obtained with error of 8.66E-15 radians.

The results for E for M (°) =120 and e = 0.9 are shown in Table 3 where the computed initial lower and upper values of $\mathbf{x_{L(0)}}$ = 2.5477169 radians = 145.95° and $\mathbf{x_{u(0)}}$ = 2.9943951radians = 171.54°.With tolerance error , ε set in the order of 10^{-15}, the results in Table 3 show that it took about 4 Regula Fasi iterations for the actual value of eccentricity (E) =1.1988490 radians = 68.68° to be obtained with error of 8.66E-15 radians.
In all, different combinations of the values of M and e give rise to different convergence cycle, as shown in Table 1, Table 2 and Table 3.

Table 1 Regula Fasi iteration results for E using the computed initial lower and maximum values of E where M (°) =1; e = 0.99; $\mathbf{x_{L(0)}}$ = 0.1179191 radians = 6.76° and $\mathbf{x_{u(0)}}$ = 1.0074533 radians = 57.72°

Cycle	x_L (radians)	x_u (radians)	X (radians)	X (degree)	f(XL)*f(x) (radians)
0	0.1179191	1.0074533	0.2021631	11.58	2.25E-04
1	0.2021631	1.0074533	0.2699947	15.47	1.62E-04
2	0.2699947	1.0074533	0.3216294	18.43	1.01E-04
3	0.3216294	1.0074533	0.3588368	20.56	5.52E-05
4	0.3588368	1.0074533	0.3844524	22.02	2.71E-05
5	0.3844524	1.0074533	0.4014926	23.00	1.22E-05
6	0.4014926	1.0074533	0.4125577	23.63	5.24E-06
7	0.4125577	1.0074533	0.4196271	24.04	2.16E-06
8	0.4196271	1.0074533	0.4240960	24.30	8.66E-07
9	0.4240960	1.0074533	0.4269018	24.46	3.43E-07
10	0.4269018	1.0074533	0.4286559	24.56	1.34E-07
11	0.4286559	1.0074533	0.4297495	24.62	5.23E-08
12	0.4297495	1.0074533	0.4304301	24.66	2.03E-08
13	0.4304301	1.0074533	0.4308533	24.68	7.84E-09
14	0.4308533	1.0074533	0.4311163	24.70	3.03E-09
15	0.4311163	1.0074533	0.4312796	24.71	1.17E-09
16	0.4312796	1.0074533	0.4313810	24.71	4.50E-10
17	0.4313810	1.0074533	0.4314440	24.72	1.74E-10
18	0.4314440	1.0074533	0.4314831	24.72	6.69E-11
19	0.4314831	1.0074533	0.4315073	24.72	2.58E-11
20	0.4315073	1.0074533	0.4315224	24.72	9.93E-12
21	0.4315224	1.0074533	0.4315317	24.72	3.83E-12
22	0.4315317	1.0074533	0.4315375	24.72	1.47E-12
23	0.4315375	1.0074533	0.4315411	24.72	5.68E-13
24	0.4315411	1.0074533	0.4315434	24.72	2.19E-13
25	0.4315434	1.0074533	0.4315447	24.72	8.43E-14
26	0.4315447	1.0074533	0.4315456	24.72	3.25E-14
27	0.4315456	1.0074533	0.4315461	24.72	1.25E-14
28	0.4315461	1.0074533	0.4315465	24.72	4.82E-15

Table 2 Regula Fasi iteration results for E using the computed initial lower and maximum values of E where M (°) = 42; e = 0.5; $x_{L(0)}$= 1.1941066radians = 68.41° and $x_{u(0)}$ = 1.2330383 radians = 70.64°

Cycle	x_L (radians)	x_u (radians)	X (radians)	X (degree)	f(XL)*f(x) (radians)
0	1.1941066	1.2330383	1.1988030	68.68	1.46E-07
1	1.1988030	1.2330383	1.1988485	68.68	1.37E-11
2	1.1988485	1.2330383	1.1988490	68.68	1.28E-15
3	1.1988490	1.2330383	1.1988490	68.68	1.20E-19
4	1.1988490	1.2330383	1.1988490	68.68	1.12E-23
5	1.1988490	1.2330383	1.1988490	68.68	1.03E-27

Table 3 Regula Fasi iteration results for E using the computed initial lower and maximum values of E where M (°) = 120; e = 0.9; $x_{L(0)}$= 2.5477169 radians = 145.95° and $x_{u(0)}$ = 2.9943951radians = 171.54°

Cycle	x_L (radians)	x_u (radians)	X (radians)	X (degree)	f(XL)*f(x) (radians)
0	2.5477169	2.9943951	2.5751726	147.53	1.09E-04
1	2.5751726	2.9943951	2.5763569	147.60	1.99E-07
2	2.5763569	2.9943951	2.5764068	147.60	3.54E-10
3	2.5764068	2.9943951	2.5764089	147.60	6.27E-13
4	2.5764089	2.9943951	2.5764090	147.60	1.11E-15
5	2.5764090	2.9943951	2.5764090	147.60	1.97E-18
6	2.5764090	2.9943951	2.5764090	147.60	3.49E-21
7	2.5764090	2.9943951	2.5764090	147.60	6.18E-24

4. CONCLUSION

Application of Regula–Falsi iteration algorithm in the computation of the eccentricity anomaly is presented. The analytical models for the selection of the two initial guess values were presented along with the flowchart for the applied Regula–Falsi method. Some sample computations of eccentricity anomaly were conducted with different combinations of the values of eccentricity and mean anomaly. The convergence cylcle of the Regula–Falsi algorithm was noted in each case. In all, different combinations of the values of M and e give rise to different convergence cycle.

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